Kronecker Delta Function δ_{ij} and Levi-Civita (Epsilon) Symbol ε_{ijk}

1. **Definitions**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \varepsilon_{ijk} = \begin{cases} +1 & \text{if } \{ijk\} = 123,\ 312, \text{ or } 231 \\ -1 & \text{if } \{ijk\} = 213,\ 321, \text{ or } 132 \\ 0 & \text{all other cases (i.e., any two equal)} \end{cases}$$

- So, for example, $\varepsilon_{112} = \varepsilon_{313} = \varepsilon_{222} = 0$.
- The +1 (or *even*) permutations are related by rotating the numbers around; think of starting with 123 and moving (in your mind) the 3 to the front of the line, to get 312. Do it again with the 2 and you get 231. The −1 (or *odd*) permutations starting with 213 are related to each other the same way; they are related to 123 by interchanging just two of the numbers (e.g., switch the 1 and 3 to get 321).

2. **Applying δ_{ij} and ε_{ijk} to Vectors in Cartesian coordinates**

- Instead of using x, y, and z to label the components of a vector, we use 1, 2, 3.
- Then the letters i, j, k, ... can be used as summation variables, running from 1 to 3. (We could use any other letters, like a, b, ...; it is merely a convention.)
- Don't confuse the use of the dummy summation variables i, j, k, each of which can be 1, 2, or 3, with the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. These are two independent notations!
- The dot product of two vectors $\mathbf{A}\cdot\mathbf{B}$ in this notation is

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\delta_{ij}\ .$$

Note that there are nine terms in the final sums, but only three of them are non-zero.

- The i^{th} component of the cross produce of two vectors $\mathbf{A}\times\mathbf{B}$ becomes

$$(\mathbf{A}\times\mathbf{B})_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k\ .$$

Again, there are nine terms in the sum, but this time only two of them are non-zero. Note also that this expression summarizes three equations, namely for $i = 1, 2, 3$.

3. **Einstein Summation Convention**

- We might notice that the summations in the expressions for $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$ are redundant, because they only appear when an index like i or j appears twice on one side of an equation. So we can omit them. Thus

$$\sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\delta_{ij} \longrightarrow A_{ij}\delta_{ij} \qquad \text{and} \qquad \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}A_jB_k \longrightarrow \varepsilon_{ijk}A_jB_k\ .$$

- Rules: If an index appears (exactly) twice, then it is summed over and appears only on one side of an equation. A single index (called a *free index*) appears once on each side of the equation. So

$$\textit{Valid:}\ A_i = A_j\delta_{ij}\,, \quad B_k = \varepsilon_{ikl}A_iC_l \qquad\qquad \textit{Invalid:}\ A_i = B_iC_i\,, \quad A_i = \varepsilon_{ijk}B_iC_j\ .$$

- When you have a Kronecker delta δ_{ij} and one of the indices is repeated (say i), then you simplify it by replacing the other i index on that side of the equation by j and removing the δ_{ij}. For example:

$$A_j\delta_{ij} = A_i\,, \qquad B_{ij}C_{jk}\delta_{ik} = B_{kj}C_{jk} = B_{ij}C_{ji}$$

  Note that in the second case we had two choices of how to simplify the equation; use either one!

- The triple or box product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ can be written

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \varepsilon_{ijk}A_iB_jC_k = \varepsilon_{kij}A_iB_jC_k = \varepsilon_{kij}C_kA_iB_j = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})\ ,$$

  where we've used the properties of ε_{ijk} to prove a relation among triple products with the vectors in a different order.

- A useful identity:

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\ .$$

4. **Example: Proving a Vector Identity**

- We'll write the i^{th} Cartesian component of the gradient operator $\boldsymbol{\nabla}$ as ∂_i.

- Let's simplify $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x}))$. We start by considering the i^{th} component and then we use our expression for the cross product:

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j(\boldsymbol{\nabla}\times\mathbf{A})_k\ .$$

  Next we replace the remaining cross product, making sure to introduce new dummy summation variables l and m:

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = \varepsilon_{ijk}\partial_j\varepsilon_{klm}\partial_lA_m = \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_lA_m\ .$$

  (The partial derivatives act only on the components of $\mathbf{A}$, so we can pull out the ε's.) We rotated the indices in one of the ε's in the last step so that we can now directly apply our useful identity (and simplify):

$$(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}))_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\partial_j\partial_lA_m = \partial_m\partial_iA_m - \partial_l\partial_lA_i = \partial_i(\partial_mA_m) - (\partial_l\partial_lA)_i$$

  or, finally,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}\ .$$