Relativity
(MTH6132)

Course notes

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Preface

These are the notes for the course of Relativity (MTH6132) I lectured in the Semester A of 2011 (October-December 2010) at the School of Mathematical Sciences of Queen Mary, University of London. These notes are mostly based on handwritten notes I have inherited from Prof. Reza Tavakol. Of course, several parts of the notes have been adapted to my particular taste and understanding of the subject. In any case, any typos, omissions or misrepresentations are entierely my responsability.

The present course on Relativity is aimed at the particular characteristics of our students at the School of Mathematical Sciences. In particular it assumed very little Physical background. Hence, a certain amount of time is spent presenting the underlying assumptions and experimental motivation for such a theory. It also assumes very little from the mathematical side. All the necessary ideas from Differential Geometry and tensors are provided within.

The course is quite an ambitious one. It begins with Special Relativity, then moves to Differential Geometry and finally (in the last third) it provides an introduction to General Relativity. Due to time constraints, there are some clear omissions in the choice of topics. In particular, in the chapter on Special Relativity it would be desirable to have a discussion of the Maxwell equations. In the chapter on General Relativity, the discussion is restricted to the vacuum field equations. There is little mention of the field equations with matter. Also, it would be desirable to have a discussion of the Friedman Cosmological models. The discussion of these topics would require at least a couple of weeks more, and may also involve the reorgainsation of some of the topics discussed in the mathematical background. I do not discard the possibility of carrying out such a revision next time I lecture the course.

Chapter 1

Introduction

1.1 What is Relativity?

The term Relativity encompasses two physical theories proposed by Einstein[1]. Namely, *Special Relativity* and *General Relativity*. However, as we will see, the word relativity is also used in reference to *Galilean Relativity*[2]. The term *Theory of Relativity* was first coined by Max Planck[3] in 1906 to emphasize how a theory devised by Einstein in 1905 —what we now call Special Relativity— uses the *Principle of Relativity*.

1.1.1 Special Relativity?

Special Relativity is the physical theory of the measurement in *inertial frames of reference*. It was proposed in 1905 by Albert Einstein in the article *On the Electrodynamic of moving bodies* (*Zur Elektrodynamik bewegter Körper*, Annalen der Physik **17**, 891 (1905)). It generalises Galileo's Principle of Relativity —all motion is relative and that there is no absolute and well-defined state of rest. Special Relativity incorporates the principle that the speed of light is the same for all inertial observers, regardless the state of motion of the source. The theory is termed *special* because it only applies to the special case of inertial reference frames —i.e. frames of reference in uniform relative motion with respect to each other. Special Relativity predicts the equivalence of matter and energy as expressed by the formula

$$E = mc^2.$$

Special Relativity is a fundamental tool to describe the interaction between elementary particles, and was widely accepted by the Physics community by the 1920's.

1.1.2 General Relativity?

General Relativity is the geometric theory of *gravitation* published by Albert Einstein in 1915 in the article *The field equations of Gravitation* (*Die Feldgleichungen der Gravitation*, Sitzungsberichte der Preussischen Akademie der Wisenschaften zu Berlin 884). It generalises Special relativity and Newton's law of universal gravitation, providing a unified description of gravity as the manifestation of the curvature of spacetime. The theory is *general* because it applies the Principle of Relativity to any frame of reference so as to handle general coordinate transformations. From General Relativity it follows that

[1] Albert Einstein (1879-1955). Physicist of German origin. Died in the USA.

[2] Galileo Galilei (584-1642) . Italian physicist, mathematician and astronomer.

[3] Max Planck (1858-1947). German physicist.

Special Relativity still applies locally. The domain of applicability of General Relativity is in Astrophysics and Cosmology. More recently, the Global Positioning System (GPS) requires of General Relativity to function accurately! Contrary to Special Relativity, General Relativity was not widely accepted until the 1960's.

1.2 Pre-relativistic Physics

1.2.1 Galilean Relativity

In order to study General Relativity one starts discussing Special Relativity. To this end, it is important to briefly look at pre-relativistic Physics to see how Special Relativity arose.

The starting point of Special Relativity is the study of motion. For this one needs the following ingredients:

- **Frames of reference.** These consist of an origin in space, 3 orthogonal axes and a clock.
- **Events.** This notion denotes a single point in space together with a single point in time. Thus, events are characterised by 4 real numbers: an ordered triple (x, y, z) giving the location in space relative to a fixed coordinate system and a real number giving the Newtonian time. One denotes the event by $E = (t, x, y, z)$.

There are an infinite number of frames of reference. Motion relative to each frame looks, in principle, different. Hence, it is natural to ask: is there a subset of these frames which are in some sense simple, preferred or natural? The answer to this question is yes. These are the so-called *inertial frames*. In an inertial frame an isolated, non-rotating, unaccelerated body moves on a straight line and uniformly.

Inertial frames are not unique. There are actually an infinite number of these. This raises the question: can one tell in which inertial frame are we in? It turns out that within the framework of Newtonian Mechanics this is not possible. More precisely, one has the following:

Galilean Principle of Relativity. Laws of mechanics cannot distinguish between inertial frames. This implies that there is no absolute rest. In other words, the laws of Mechanics retain the same form in different inertial frames.

In this sense, Relativity predates Einstein.

1.2.2 Laws of Newton

The three Laws of Newtonian Mechanics[4] are:

(1) Any material body continues in its state of rest or uniform motion (in a straight line) unless it is made to change the state by forces acting on it. This principle is equivalent to the statement of existence of inertial frames.

(2) The rate of change of momentum is equal to the force.

(3) Action and reaction are equal and opposite.

[4]Isaac Newton (1643-1727). English physicist and mathematician.

These laws or principles, together with the following fundamental assumptions (some of which are implicitly assumed in Newton's laws) amount to the *Newtonian framework*:

(A1) Space and time are continuous —i.e. not discrete. This is necessary to make use of the Calculus.

(A2) There is a universal (absolute) time. Different observers in different frames measure the same time. In fact, Newton also regarded space to be absolute as well. However, the absoluteness of space is not necessary for the development of the Newtonian framework, as space intervals turn out to be invariant under Galilean transformations. Historically, Newton demanded this for subjective reasons.

(A3) Mass remains invariant as viewed from different inertial frames.

(A4) The Geometry of space is Euclidean. For example, the sum of angles in any triangle equals 180 degrees.

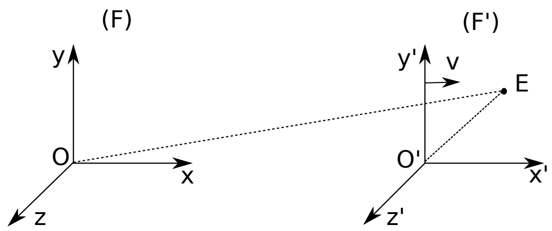
(A5) There is no limit to the accuracy with which quantities such as time and space can be measured.

As it will be seen in the sequel, Assumptions 2 and 3 are relaxed in Special Relativity while Assumption 4 is relaxed in General Relativity. Assumption 5 is relaxed in Quantum Mechanics —not to be discussed in the course. Presumably Assumption 1 will be relaxed in Quantum Gravity!

1.2.3 Galilean transformations

Galilean transformations tell us how to transform from one inertial frame to another.

Consider two inertial frames: $F(x, y, z, t)$ and $F'(x', y', z', t')$ moving with velocity $\underline{v}$ relative to one another in *standard configuration* —that is, F' moves along the x axis of the frame F with uniform speed v; all axes remain parallel. See the figure:



Now, suppose that at a given moment of time t, an event E is specified by coordinates (t, x, y, z) and (t', x', y', z') relative to the frames F and F', respectively. Let the origins O and O' coincide at $t = 0$. From the figure one sees that

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t, \tag{1.1}$$

or more compactly (recall that in general $\underline{v} = (v_x, v_y, v_z)$, but here $v_y = v_z = 0$):

$$\underline{r}' = \underline{r} - \underline{v}t.$$

In general, if the coordinate axes are not in standard configuration and the origins O and O' of the coordinate axes do not coincide, then the general form of the transformation takes the form:

$$\underline{r}' = R\underline{r} - \underline{v}t + \underline{d},$$

where R is the rotation matrix aligning the axes of the frames and $\underline{d}$ is the distance between the origins at $t = 0$. Note that the general transformation is linear, so that F' is inertial if F is. The most general transformation would also include

$$t' = t + \tau$$

where τ is a real constant.

These transformations form a 10-parameter group (1 for τ, 3 for $\underline{v}$, 3 for $\underline{d}$, and 3 for R). The group property implies that the composition of two Galilean transformations is a Galilean transformation, and that given a Galilean transformation there is always an inverse transformation. The Galilean transformations restricted to standard configurations form a 1-parameter subgroup of this group, with v as variable.

1.2.4 Invariance of Newton's laws under Galilean transformations

Important for the sequel is the notion of invariance. Invariance referes to properties of a system that remain unchanged under a particular type of transformations. For example, $\underline{a} = \underline{b}$ as a vector equation is invariant under a change in coordinate system. However, the particular values of components of the vectorial equation do change. We will see more about this in the next chapter!

We will see that the laws of Mechanics keep the same form as we go from one inertial frame to another —i.e. under Galilean transformations. To see this, let the position of a particle P be specified by $\underline{r} = \underline{r}(t)$ relative to a frame F. The motion relative to F' is given by equation (1.1). Differentiating both sides twice with respect to t (notice that $t = t'$) gives:

$$\underline{v}' = \underline{v} - \underline{V}, \quad \underline{a}' = \underline{a},$$

where

$$\underline{v} = \frac{\mathrm{d}\underline{r}}{\mathrm{d}t}, \quad \underline{a} = \frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2},$$

are, respectively, the velocity and acceleration of the particle.

Now, the First and Third Laws are invariant as the former involves inertial frames and the latter involves accelerations which are invariant. It remains to show that the Second Law (the fundamental equation of Newtonian Mechanics)

$$m\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = m\underline{a} = \underline{f} \tag{1.2}$$

is invariant as we go from one inertial frame to another.

To show the invariance of (1.2) recall that $\underline{a}' = \underline{a}$ and m remains invariant (by assumption) so that one only needs to show that $\underline{f}$ remains invariant as we go from F to F'. To do this, recall that generally $\underline{f}$ takes the form $\underline{f} = \underline{f}(\underline{r}, \underline{v}, t)$ where usually $\underline{r}$ and $\underline{v}$ are the relative distance and the relative velocity between two bodies. One can verify that the relative distances and velocities remain invariant. That is,

$$\underline{v}_2' - \underline{v}_1' = \underline{v}_2 - \underline{v}_1, \quad \underline{r}_2' - \underline{r}_1' = \underline{r}_2 - \underline{r}_1.$$

This implies that $\underline{f}$, and hence the Second Law remains invariant under changes in the inertial frames.

This discussion amounts to a form of self-consistency, in the sense that Physics, when confined to Newtonian Mechanics, satisfies the Galilean Principle of Relativity.

1.2.5 Electromagnetism

Special Relativity arises from the tension between Newtonian Mechanics with the other great physical theory of the 19th century —Electromagnetism. The fundamental laws of Electromagnetism are the so-called *Maxwell equations*[5] :

$$
\begin{aligned}
&\nabla \cdot \underline{D} = \rho, \\
&\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}, \\
&\nabla \cdot \underline{B} = 0, \\
&\nabla \times \underline{H} = \underline{j} - \frac{\partial \underline{D}}{\partial t},
\end{aligned}
$$

where $\underline{B}$ is the magnetic induction, $\underline{E}$ the electric field, $\underline{H}$ the magnetic field, $\underline{D}$ the electric displacement, $\underline{j}$ the electric current and ρ the electric charge.

It can be shown that these equations predict the existence of electromagnetic waves for $\underline{E}$ and $\underline{H}$ in the form

$$
\nabla^2 \underline{E} = \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2}, \quad \nabla^2 \underline{H} = \frac{1}{c^2}\frac{\partial^2 \underline{H}}{\partial t^2},
$$

where c is the speed of propagation of the waves. These electromagnetic waves were soon identified with the propagation of light.

We recall that speed travels with a speed of $c \approx 3 \times 10^8 m/s$. This was first measured by Rømer [6] in 1675 by studying the delay in the appearance of moons of Jupiter.

Within the Newtonian framework, the Maxwell equations give rise to two problems:

(1) With respect to which system of reference is the speed of light c is measured? First, it was assumed that the absolute space of Newton —the so-called *ether*— was the medium in (and relative to) which light moved. However, attempts at detecting the effects of Earth's motion on the velocity of light —the so-called terrestrial ether drift— all failed. The most important of these was the *Michelson-Morley experiment*[7]. This gave a null result.

(2) It is easy to show that Maxwell's equations and the wave equation do not remain invariant under Galilean transformations.

These problems gave to a crisis in the 19th century Physics. Three scenarios were put forward to resolve the tension. These were:

(i) Maxwell's equation were incorrect. The correct laws of Electromagnetism would remain invariant under Galilean transformations.

(ii) Electromagnetism had a preferred frame of reference —that of ether.

(iii) There is a Relativity Principle for the whole of Physics —Mechanics and Electromagnetism. In that case the laws of Mechanics need modification.

Now, Electromagnetism was very successful and have a very strong predictive power. There was no experimental support for (ii). Hence the point of view (iii) was adopted by Einstein. His resolution of the tension between Mechanics and Electromagnetism came to be known as Special Relativity.

[5] James C. Maxwell (1831-1879). Scottish mathematician.
[6] Ole C. Rømer (1644-1710). Danish astronomer.
[7] Albert Michelson (1852-1931). Edward Morley (1838-1923). American physicists.

Chapter 2

Special Relativity

The contradiction brought about by the development of Electromagnetism gave rise to a crisis in the 19th century that Special Relativity resolved.

2.1 Einstein's postulates of Special Relativity

(i) There is no ether (there is no absolute system of reference).

(ii) The laws of Nature have the same form in all inertial frames (Einstein's principle of Relativity)

(iii) The velocity of light in empty space is a universal constant, i.e. same for all observers and light sources, independent of their motion —Michelson & Morley's result is promoted to an axiom.

Note that postulate (iii) is clearly incompatible with Galilean transformations which imply $c' = c - v$. Because of this the Galilean transformations need modification. This leads to the *Lorentz transformations*.

2.2 Spacetime pictures

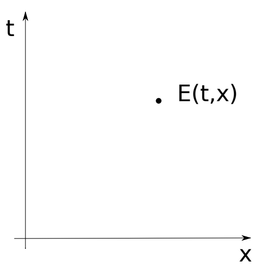
This is a very useful way to think.

2.2.1 Some definitions

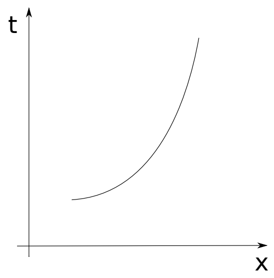
Spacetime. Defined as the set of 4 reals (t, x, y, z).

For simplicity (in order to be to visualise) confine ourselves to 2 dimensions: one space and one time coordinates.

Event. Represented by a point in spacetime: i.e. $E(t, x, y, z)$ or $E(t, x)$.

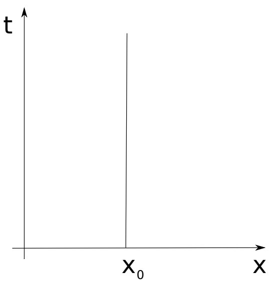


Wordline. Defined as the set of all points that the trajectory of a particle follows in spacetime.



2.2.2 Examples

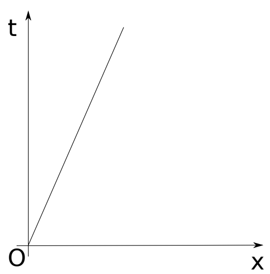
- Worldline of a particle stationary at $x = x_0$.



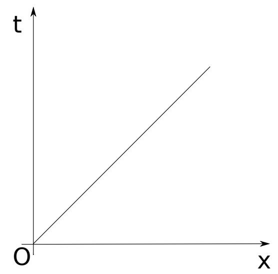
- Worldline of a particle moving with uniform velocity v and passing through O at $t = 0$ is straight line:

$$x = vt \quad \text{so that} \quad t = \frac{1}{v}x.$$

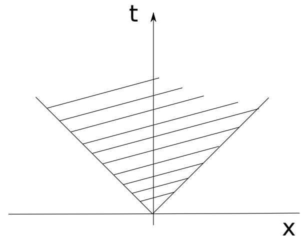
  Therefore the slope of of the line is given by $1/v$.



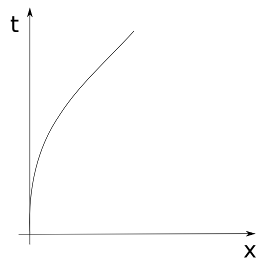
- The worldline of a light ray is a straight line with slope equal to $1/c$. In practice we shall usually choose $c = 1$ so that the slope is equal to 1.



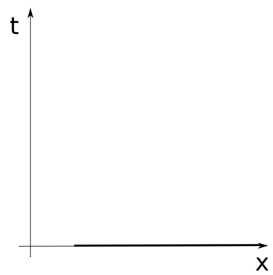
Note. All uniformly moving particles have worldlines which are straight lines with slopes *bigger* than $1/c$ or bigger than 1 if $c = 1$. Therefore they all lie in the shaded region of the figure.



- The worldlines of accelerating bodies are curved. For example, for a uniformly accelerated body from rest one has that initially the worldline is tangent to the t. The upper bound for v is c. The slope of the asymptotic motion is $1 (= 1/c)$. This situation will be analysed in detail later on.

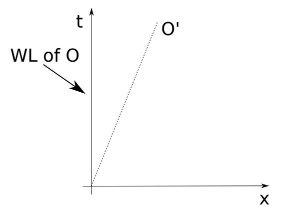


- The worldlines of instantaneous travel is a horizontal line —however, this is forbidden within the framework of Special Relativity.



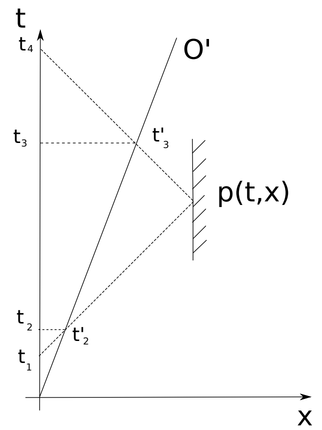
2.3 Lorentz transformations (LT)

Consider two frames F and F' moving in standard configuration —i.e. O' moves with speed v along the x-axis relative to O. The worldline of O' in the frame is given as in the figure:



Let observers O and O' carry clocks measuring t and t' respectively such that when O' is at (t, vt) according to O, the clock at O' registers $t' = \beta t$, where β may be a function of v —in this sense β carries all the effect that the motion has on t. Note also that $\beta = 1$ for Galilean transformations.

Now consider a light ray emitted by O at $t = t_1$, travelling via O', being reflected at $p(t, x)$ and received by O at $t = t_4$ —i.e. a round trip.



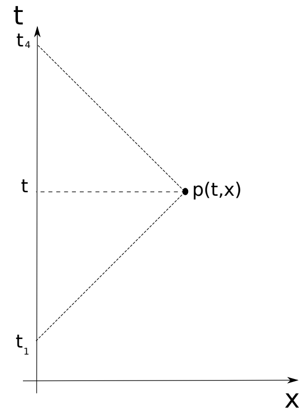
We want to relate the coordinates of the event at p relative to the frames F and F'.

In line with Einstein's postulates assume that the speed of light is c for both O and O'. From the perspective of O the distance and time may be fixed using the so-called *radar convention*:

$$x = \tfrac{1}{2}c(t_4 - t_1), \quad t = \tfrac{1}{2}(t_4 + t_1)$$

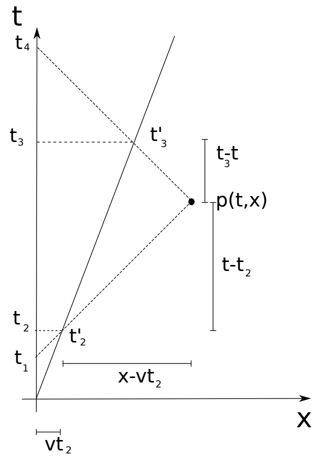
so that

$$x = c(t - t_1) = c(t_4 - t). \tag{2.1}$$



Similarly,

$$x - vt_2 = c(t - t_2), \tag{2.2}$$
$$x - vt_3 = c(t_3 - t). \tag{2.3}$$



Now, equations (2.2) and (2.3) imply, respectively

$$t_2 = \frac{ct - x}{c - v}, \quad t_3 = \frac{ct + x}{c + v}.$$

The corresponding times as measured by O' are:

$$t'_2 = \beta t_2 = \beta \left(\frac{ct - x}{c - v} \right), \tag{2.4a}$$
$$t'_3 = \beta t_3 = \beta \left(\frac{ct + x}{c + v} \right), \tag{2.4b}$$

where it has been used that $t' = \beta t$. Therefore, the time and location of $p(t, x)$ as measured by O' is (using again the radar convention) is given by:

$$x' = \tfrac{1}{2} c(t'_3 - t'_2) = \frac{\beta c^2 (x - vt)}{c^2 - v^2}, \tag{2.5a}$$
$$t' = \tfrac{1}{2}(t'_3 + t'_2) = \frac{\beta(c^2 t - vx)}{c^2 - v^2}, \tag{2.5b}$$

where equations (2.4a) and (2.4b) have been used to obtain the second equalities in the last pair of equations.

Note. The observer O' is also assuming that the velocity of light is c. This assumption is inconsistent with the Galilean transformations.

Eliminating x between (2.5a) and (2.5b) one obtains

$$t = \frac{1}{\beta}\left(t' + \frac{vx'}{c^2}\right). \tag{2.6}$$

Now, the *Relativity principle* requires that we obtain the same result if we interchange x, x' and t, t' and let $v \to -v$. Applying this idea to equation (2.5b) and equating to (2.6):

$$t = \frac{\beta(c^2 t' + vx')}{c^2 - v^2} = \frac{1}{\beta}\left(t' + \frac{vx'}{c^2}\right), \tag{2.7}$$

so that

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{1/2}.$$

Letting $\gamma \equiv 1/\beta$, the transformation for x' can be found from (2.5a):

$$x' = \gamma(x - vt). \tag{2.8}$$

Similarly for t from equation (2.5b):

$$t' = \gamma\left(t - \frac{vx}{c^2}\right).$$

Finally, the coordinates y and z remain the same as there is no motion in these directions.

Thus, we have obtained the so-called *Lorentz transformations*:

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, z' = z. \tag{2.9}$$

The inverse transformation can be obtained by letting $x \to x'$, $t \to t'$ and $v \to -v$ to yield:

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right), \quad y = y', \quad z = z'.$$

Remark. This is the case of a more general transformation with 10 parameters. These parameters are the 3 components of the velocity, 3 components of a shift of the origin, 3 parameters of a rotation and a further parameter fixing the origin of the time. The set of these transformations forms a group. The transformation given by (2.9) is the 1-parameter subgroup of this group called the *special Lorentz group*.

2.4 Hyperbolic form of the Lorentz transformations

This a convenient representation for showing the group properties of the Lorentz transformation.

The key idea is to replace the velocity parameter v by a hyperbolic parameter α satisfies the following:

$$\cosh\alpha = \gamma, \quad \sinh\alpha = \frac{v}{c}\gamma, \quad \tanh\alpha = \frac{v}{c}.$$

We also require α and v to have the same sign as $\cosh\alpha = \cosh(-\alpha)$.

The Lorentz transformation (2.9) becomes (*hyperbolic form of the Lorentz transformation*):

$$
\begin{aligned}
x' &= x\cosh\alpha - ct\sinh\alpha, && (2.10a)\\
ct' &= -x\sinh\alpha + ct\cosh\alpha, && (2.10b)\\
y' &= y, && (2.10c)\\
z' &= z && (2.10d)\\
& && (2.10e)
\end{aligned}
$$

Adding and subtracting x' and ct' as given by (2.10a) and (2.10b) one obtains

$$
\begin{aligned}
ct' + x' &= e^{-\alpha}(ct + x), && (2.11a)\\
ct' - x' &= e^{\alpha}(ct - x), && (2.11b)
\end{aligned}
$$

where it has been used that

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}.$$

To show that the Lorentz transformations form a group one needs to show:

(i) there exists an identity element;

(ii) for every Lorentz transformation there exists an inverse;

(iii) the composition of Lorentz transformations is a Lorentz transformation and that the composition is associative.

The most convenient way to verify the latter is to use the form given by (2.11a) and (2.11b) and then check one by one:

(i) One sees that there exists an identity Lorentz transformation corresponding to v ($\alpha = 0$).

(ii) There exists an inverse Lorentz transformation with $v = -v$ ($\alpha \to -\alpha$).

(iii) Let F'' move with velocity v_2 (α_2) relative to F' and F' with velocity v_1 (α_1) relative to F —all in standard configuration.

From (2.11a) and (2.11b) one has that

$$
\begin{aligned}
ct'' + x'' &= e^{-\alpha_2}(ct' + x'),\\
ct'' - x'' &= e^{\alpha_2}(ct' - x'),\\
y'' &= y, \quad z'' = z'
\end{aligned}
$$

and

$$
\begin{aligned}
ct' + x' &= e^{-\alpha_2}(ct + x),\\
ct' - x' &= e^{\alpha_2}(ct - x),\\
y' &= y, \quad z' = z.
\end{aligned}
$$

It follows then that

$$
\begin{aligned}
&ct'' + x'' = e^{-(\alpha_1+\alpha_2)}(ct + x),\\
&ct'' - x'' = e^{(\alpha_1+\alpha_2)}(ct - x),\\
&y'' = y, \quad z'' = z',
\end{aligned}
$$

which shows that the composition of Lorentz transformations is a Lorentz transformation and since the hyperbolic parameters add, one also has the associativity.

The previous discussion allows also to discuss the Special Relativity rule for the *composition of velocities*. Since the resultant of two Lorentz transformations with parameters α_1 and α_2 is a Lorentz transformation with parameters $\alpha_1+\alpha_2$, the corresponding relation between the velocity parameter of the transformation can be easily derived from

$$
\tanh\alpha = \frac{v}{c}
$$

by recalling that

$$
\tanh(\alpha_1 + \alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 + \tanh\alpha_1 \tanh\alpha_2}.
$$

Substituting for

$$
\tanh\alpha_1 = \frac{v_1}{c}, \quad \tanh\alpha_2 = \frac{v_2}{c}, \quad \tanh\alpha_1 + \alpha_2 = \frac{v}{c}
$$

one obtains

$$
v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \tag{2.12}
$$

where v is the velocity of F'' relative to F —it represents the *relativistic sum of collinear velocities v_1 and v_2 along the x-axis.* A generalisation of this rule will be discussed later.

Remark 1. When

$$
\frac{v_1}{c} \ll 1, \quad \frac{v_2}{c} \ll 1,
$$

then equation (2.12) takes the Galilean form

$$
v = v_1 + v_2.
$$

Remark 2. Since $|\tanh\alpha| < 1$, it follows that v always satisfies $|v| < c$.

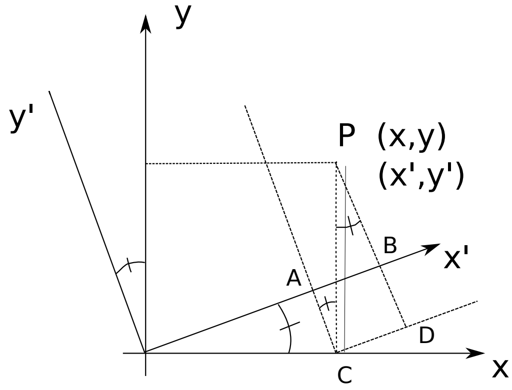
2.5 The Minkowski spacetime

There are many ways to study Special relativity. here we take the geometrical approach developed in 1908 by H. Minkoswki. This approach naturally leads to (and led Einstein!) to General Relativity.

To gain some intuition, start with the Euclidean geometry of the 2 dimensional plane and recall the transformation of coordinates corresponding to the rotation of Cartesian axes by an angle α in such a plane:

$$
\begin{aligned}
x' &= x\cos\alpha + y\sin\alpha,\\
y' &= -x\sin\alpha + y\cos\alpha,
\end{aligned}
$$

where (x, y) and (x', y') correspond to the coordinates of the point p in the two frames.



The transformation can be deduced from the diagram by observing that:

$$\begin{aligned} x' &= OA + AB = OA + CD \\ &= OC\cos\alpha + PC\sin\alpha \\ &= x\cos\alpha + y\sin\alpha \\ y' &= PB = PD - BD \\ &= PC\cos\alpha - OC\sin\alpha \\ &= -x\sin\alpha + y\cos\alpha. \end{aligned}$$

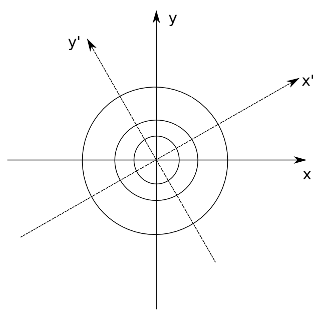
Eliminating the rotation parameter α by taking

$$\begin{aligned} x'^2 + y'^2 &= (x\cos\alpha + y\sin\alpha)^2 + (-x\sin\alpha + y\cos\alpha)^2 \\ &= x^2 + y^2. \end{aligned}$$

Letting

$$(OP) \equiv x^2 + y^2, \tag{2.13}$$

one sees that in Euclidean space, rotations leaves the distance (OP) *invariant*. Note also that the rotation leaves curves of constant distance from the origin —i.e. circles— invariant.



Analogue for Lorentz transformations. Starting from

$$\begin{aligned} ct' + x' &= e^{-\alpha}(ct + x), \\ ct' - x' &= e^{\alpha}(ct - x), \end{aligned}$$

and multiplying both sides one obtains

$$-ct^2 + x^2 = -ct'^2 + x'^2,$$

where the choice of sign in the previous equation is a convention. Furthermore, since $y' = y$ and $z' = z$ one obtains

$$-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2. \tag{2.14}$$

Alternatively, one could start from the infinitesimal version of the Lorentz transformations

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right), \quad \Delta x' = \gamma\left(\Delta x - v\Delta t\right), \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z,$$

and taking the limit in equation (2.14) one obtains

$$-c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = -c^2\mathrm{d}t'^2 + \mathrm{d}x^{2\prime} + \mathrm{d}y^{2\prime} + \mathrm{d}z^{2\prime}. \tag{2.15}$$

Therefore

$$-c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

remains invariant under Lorentz transformations (boosts).

Remark 1. The value of c is unit dependent. Often, relativists choose units (relativistic units) such that $c = 1$. That is, distance is measured in *light seconds* —the distance travelled by light in 1 second. *From now on we shall put* $c = 1$. Subsequent formulae may be put "right" dimensionally by putting the missing c's back on basis of dimensional grounds.

Remark 2. With $c = 1$ one has that equation (2.15) reduces to

$$-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

which, apart from the negative sign is very similar to the Euclidean distance in 4 dimensions

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2.$$

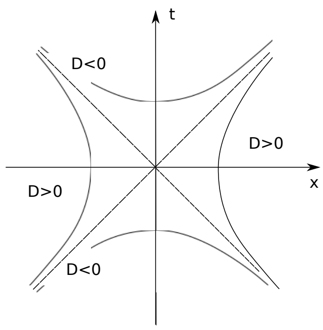
Furthermore, they both remain invariant under coordinate transformations: Lorentz transformations and rotations, respectively. This invariant quantity is called the *interval* $\mathrm{d}s^2$ (or *line element*) in a new type of geometry called the *Minkowski geometry* or *spacetime*. It is then described by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

The latter measures the "distance" between events (t, x, y, z) and $(t + \mathrm{d}t, x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ in spacetime.

Note. As opposed to Euclidean geometry, the set of points with equal distances from the origin defines a *hyperbola*:

$$x^2 - t^2 = D, \quad D \text{ a constant.}$$



2.6 Minkowski diagrams

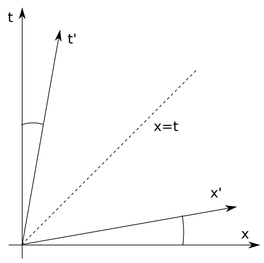
The consequence of Special Relativity are best visualised using Minkowski diagrams. These are pictures in Minkowski spacetime (usually $x-t$ pictures). As an example let us look at the positions of the x' and t' axes relative to the x and t axes.

The x' axis (i.e. $t'=0$) is given by ($c=1$):

$$t' = \gamma(t - vx), \quad \text{so that} \quad t = vx.$$

Similarly, the t' axis (i.e. $x'=0$) is given by

$$x' = \gamma(x - vt) = 0, \quad \text{so that} \quad t = \frac{1}{v}x$$



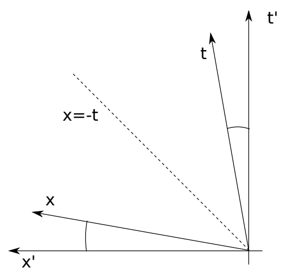
One can also ask what is seen in the reference frame F'. For this one can use the inverse Lorentz transformations

$$t = \gamma(t' + vx'), \quad x = \gamma(x' + vt').$$

The x and t axes from the point of view of the frame F' are given, respectively, by

$$t' = -vx', \quad t' = -\frac{1}{v}x.$$

Thus, the picture from F''s point of view is the following:



This picture is consistent with the Principle of Relativity —all frames of reference are equivalent and should provide an equivalent picture! We shall see further examples of this symmetry in the sequel.

2.7 Index notation

In what follows let

$$(t, x, y, z) = (x^0, x^1, x^2, x^3),$$

where the index position is a convention —more about this later. Write

$$x^a, \quad (a = 0,\ 1,\ 2,\ 3)$$

for x^0, x^1, x^2, x^3 we may write (2.15) as

$$\mathrm{d}s^2 = \sum_{a=0}^{3}\sum_{b=0}^{3} \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{2.16}$$

where η_{ab} is called the *Minkowski metric tensor* given by

$$(\eta_{ab}) \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$\eta_{11} = \eta_{22} = \eta_{33} = 1, \quad \eta_{00} = -1,$$

while all other η_{ab}'s are zero.

In order to drop clumsy summations *hereafter* we will use the so-called *Einstein summation convention*:

(i) Whenever an index is repeated (appears exactly twice) in a term, it is understood to imply summation over that index over all its permissible values. In this course lower case Latin indices $a,\ b, \ldots$ take values 0, 1, 2, 3. Hence equation (2.16) may be written

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b.$$

(ii) Repeated indices as called *dummy indices* since they may be replaced by another index (from the same alphabet!) not already used. For example:

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = \eta_{cd}\mathrm{d}x^c\mathrm{d}x^d.$$

(iii) To avoid ambiguity, no index should appear more than twice in the same expression. So

$$a_i b_i c_i$$

is not allowed!

(iv) Indices that occur only once in an expression (or terms of an equation) are called *free indices*. In an equation such indices match in every term. For example consider

$$A^i B_i C_j = D_j.$$

Notice that i is a dummy index and that j is a free index.

Examples

For simplicity in the following examples let the Latin lower case index take values 1, 2.

(1)
$$A^i B^j = A^1B^1,\ A^1B^2,\ A^2B^1,\ A^2B^2$$
as $i,\ j$ are free indices.

(2)
$$A^i B_i = \sum_{i=1}^{2} A^i B_i = A^1B_1 + A^2B_2,$$
as i is a dummy index.

(3)
$$g_{ij} = g_{11},\ g_{12},\ g_{21},\ g_{22}$$
as, again, $i,\ j$ are free indices.

(4) In Γi_{jk} all indices are free. There are 8 terms: $\Gamma^1{}_{11},\ \Gamma^1{}_{12};\ldots.$

(5) In $R^i{}_{jkl}$ all indices are free and there are 16 terms: $R^1_{111},\ R^1{}_{112},\ R^1{}_{122},\ldots$

(6)
$$\Gamma^i{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = \Gamma^i{}_{lm}\frac{\mathrm{d}x^l}{\mathrm{d}s}\frac{\mathrm{d}x^m}{\mathrm{d}s}$$
as $l,\ m$ are dummy indices while i is free.

(7) $x_a y_b z^b = z_a y_c y^c$.

(8) $g_{ij}\mathrm{d}x^i\mathrm{d}x^j = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = g_{11}(\mathrm{d}x^1)^2 + g_{12}\mathrm{d}x^1\mathrm{d}x^2 + g_{21}\mathrm{d}x^2\mathrm{d}x^1 + g_{22}(\mathrm{d}x^2)^2$.

2.8 4-vectors in Special Relativity

In order to write Newton's laws in the Minkoswki spacetime, we require 4-vectors.

In analogy with 3-vectors (which are invariant under the change of coordinates) we define 4-vectors in the Minkowski 4-dimensional geometry in such a way that the resulting calculus will have equations invariant under Lorentz transformations (boosts).

4-vectors

A 4-vector is a set of four ordered real numbers which transform in exactly the same manner as do (t, x, y, z) under Lorentz transformations.

Denote 4-vectors by overlines; as opposed to 3-vectors denoted by underlines.

In index notation
$$\bar{A} = (A^i) \equiv (A^0, A^1, A^2, A^3).$$
The Lorentz transformation relating A^i to A'^i may be written as
$$A'^i = L^i{}_j A^j \quad \text{summation over } j$$

where $L^i{}_j$ is the Lorentz transformation matrix defined as

$$(L^i{}_j) \equiv \begin{pmatrix} L^0{}_0 & L^0{}_1 & L^0{}_2 & L^0{}_3 \\ L^1{}_0 & L^1{}_1 & L^1{}_2 & L^1{}_3 \\ L^2{}_0 & L^2{}_1 & L^2{}_2 & L^2{}_3 \\ L^3{}_0 & L^3{}_1 & L^3{}_2 & L^3{}_3 \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Check:

$$A'^0 = (\gamma, -v\gamma, 0, 0) \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \gamma(A^0 - vA^1).$$

So it transforms like x^0.

Norm or magnitude of a 4-vector

It is defined by

$$|\bar{A}|^2 \equiv (A^1)^2 + (A^2)^2 + (A^3)^2 - (A^0)^2 = \eta_{ab} A^a A^b, \tag{2.17}$$

which in analogy with the invariance of

$$x^2 + y^2 + z^2 - t^2 = |\bar{x}|^2, \quad \bar{x} = (t, x, y, z),$$

is invariant.

Exercise: Show by direct substitution that the norm of a 4-vector is invariant. One has that

$$\begin{aligned} A'^0 &= \gamma(A^0 - vA^1), \\ A'^1 &= \gamma(A^1 - vA^0), \\ A'^2 &= A^2, \quad A'^3 = A^3. \end{aligned}$$

Hence,

$$\begin{aligned} -(A'^0)^2 + (A'^1)^2 &= \gamma^2(A^1)^2 + \gamma^2 v^2 (A^0)^2 - 2\gamma v A^1 A^0 - \gamma^2 (A^0)^2 - \gamma^2 v^2 (A^1)^2 + 2\gamma v A^0 A^1, \\ &= \gamma^2 (A^1)^2 (1 - v^2) - \gamma^2 (A^0)^2 (1 - v^2), \\ &= (A^1)^2 - (A^0)^2. \end{aligned}$$

Remark. Because of the negative sign in (2.17), the norm of a vector does not have to be positive! A 4-vector $\bar{A}$ is said to be:

- **timelike** if $|\bar{A}|^2 < 0$,
- **spacelike** if $|\bar{A}|^2 > 0$,
- **null** if $|\bar{A}|^2 = 0$.

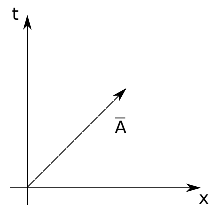
In Minkowski spacetime a null vector need not be a zero vector whose components are zero! Only in a space in which the norm is positive definite, it is true that $|A|^2 = 0$ implies $A = 0$.

Example: Show that $\bar{A} = (1, 1, 0, 0)$ is a null vector. A direct computation gives

$$|\bar{A}|^2 = -(A^0)^2 + (A^1)^2 = -1 + 1 = 0.$$

Similarly for

$$(1, -1, 0, 0), \quad (1, 0, 1, 0), \quad (1, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), \quad \text{etc.}$$

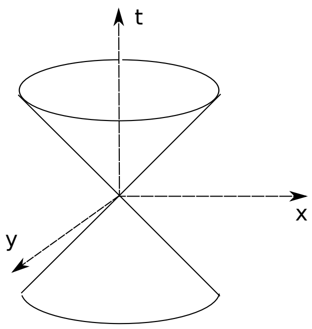


The light cone

Take the hypersurface in spacetime defined by $|\bar{A}|^2 = 0$ which for $\bar{A} = (t, x, y, z)$ so that

$$x^2 + y^2 + z^2 - t^2 = 0.$$

This is said to define a *light cone* at the origin, because all lights rays emitted at $t = 0$ at origin lie on the cone $x^2 + y^2 + z^2 = c^2 t^2$. Suppressing 1-space dimension one has the following figure:



Similarly, $|\bar{A}|^2 < 0$ implies $x^2 + y^2 + z^2 - t^2 < 0$ corresponds to the interior of the light cone containing the t axis. Similarly, $|\bar{A}|^2 > 0$ corresponds to the exterior of the cone.

Scalar product

The scalar product of two 4-vectors $\bar{A}$, $\bar{B}$ is defined by

$$\bar{A} \cdot \bar{B} = \eta_{ab} A^a B^b = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3.$$

Notice that as a consequence of this definition $|\bar{A}|^2 = \bar{A} \cdot \bar{A}$.

Example: Prove that $\bar{A} \cdot \bar{B}$ is invariant under Lorentz transformations: start with

$$(\bar{A} + \bar{B}) \cdot (\bar{A} + \bar{B}) = |\bar{A}|^2 + |\bar{B}|^2 + 2\bar{A} \cdot \bar{B},$$

and note that $|\bar{A}|^2$, $|\bar{B}|^2$ and $|\bar{A} + \bar{B}|^2$ are all invariants. Hence, so is $\bar{A} \cdot \bar{B}$.

Orthogonality

Two vectors are called orthogonal if $\bar{A} \cdot \bar{B} = 0$.

Note 1. Because of the nature of the Minkowski geometry, two orthogonal 4-vectors do not appear orthogonal graphically.

Note 2. Null vectors are orthogonal to themselves ($\bar{A} \cdot \bar{A} = 0$)!

Basic 4-vectors

In any frame F, there exist 4 *basic vectors*

$$\bar{e}_0 = (1, 0, 0, 0), \quad \bar{e}_1 = (0, 1, 0, 0), \quad \bar{e}_2 = (0, 0, 1, 0), \quad \bar{e}_3 = (0, 0, 0, 1),$$

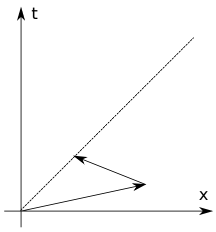
in terms of which any 4-vector $\bar{A}$ may be expressed:

$$\bar{A} = A^i e_i = A^0 e_0 + A^1 e_1 + A^2 e_2 + A^3 e_3,$$

where A^i are the components of $\bar{A}$

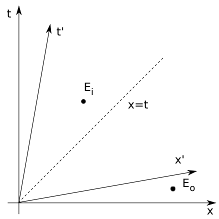
One can add and subtract 4-vectors pictorially like is done for 3-vectors.

Example: With the help of a sketch convince yourself that the sum of two timelike or spacelike vectors or the sum of a timelike and a spacelike vector can be null!



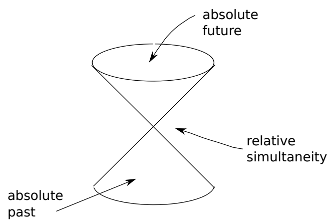
2.9 A brief discussion of causality

In what follows we discuss some consequences of the x-dependence of the the Lorentz transformation of time.



(1) Any event E_i inside the light cone occurring after O from the perspective of F will also occur after O from the perspective of F' no matter how fast F' moves with respect to F so long as $v \leq c$. An event E_0 outside the light cone and occurring after O from the point of view of F could occur before O from the point of view of F'. Therefore, outside the future (and similarly the past) light cone of O there exists no ordered time sense of events.

Given any point O, the spacetime is divided up into *the absolute past of* O (the past light cone at O) and the *absolute future of* O (the future light cone at O) and a region (spacelike) know as the *region of relative simultaneity*.



(2) For invariance of causality, interactions must take place at speeds less than c. To see this, consider a process in which an event E_1 causes an event E_2 at super-light speed $u > c$ relative to some frame F. Choose coordinates in F such that E_1 and E_2 occur on the x-axis and let their space and time separation $\Delta x > 0$, $\Delta t > 0$ (i.e. E_1 precedes E_2). Now, in frame F' moving with with velocity v relative to F we have:

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \gamma \Delta t\left(1 - \frac{uv}{c^2}\right)$$

where

$$u = \frac{\Delta x}{\Delta t}$$

is the speed of propagation. Now, for

$$\frac{c^2}{u} < v < c$$

we would have $\Delta t' < 0$ so that in F' the event E_2 precedes E_1 —i.e. cause and effect are reversed or we have information from receiver to transmitter!

2.10 Clocks and rods in relativistic motion

We now consider the effects of uniform motion on clocks and rods.

2.10.1 Time dilation

Consider F and F' in standard configuration. Let a standard clock be at rest in F' (at $x = x_0$) and consider two events in this clock at times t'_1 and t'_2. Let also

$$\Delta t' = t'_2 - t'_1.$$

In order to find the interval Δt as measured by F, recall that

$$\Delta t = \gamma(\Delta t' + v\Delta x').$$

However, $\Delta x' = 0$ as $x'_2 = x'_1 = x_0$. Hence one obtains

$$\Delta t = \gamma \Delta t',$$

Since

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1,$$

one finds that the interval as measured by F is longer.

There is a symmetry! Both observers say the same thing about each other!

2.10.2 Length contraction

This is also called the (Lorentz-Fitzgerald contraction). Consider F and F' in standard configuration. Let a rod of length $\Delta x'$ be placed at rest along the x'-axis of F'. To find the length as measured in F, we must measure the distance between the two ends of the rod simultaneously in F. Consider two events occurring simultaneously at the end points of the rod in F. Therefore one has $\Delta t = 0$. Now, using

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$

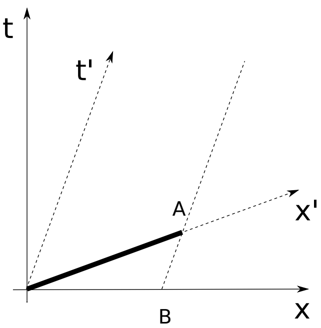
one finds that

$$\Delta x' = \gamma \Delta x, \quad \text{or } \Delta x = \frac{1}{\gamma}\Delta x'.$$

Accordingly, the length of the rod in the direction of motion as measured by F is reduced by a factor of $(1 - v^2)^{1/2}$.

Geometrically:

F measures the distance between the two ends of the rod at $t = 0$, i.e. F measures OB, while F' measures OA.



2.11 Paradoxes

These arise from an incautious view of the situation, and the fact that simultaneity means different things to different observers.

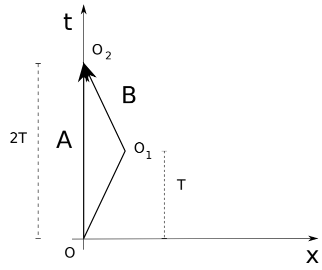
The twin paradox

Consider a pair of twins A and B. Let A be stationary at origin of F whereas B moves with sped v for a time T and then with speed $-v$ for equal time and returns to A's position. The total elapsed time as measured by A is $2T$. Because of time dilation, the time as measured by B is

$$\frac{2T}{\gamma} < 2T.$$

Therefore, when twins reach the point $(0, 2T)$ in A's frame A is older than B.

The "paradox": cannot B say with equal right that it was she/he who remained where she/he was while A went on a round trip and that A should, consequently, be the younger when they meet?



Answer: No, since there is no symmetry! The twin A remained in the same inertial frame, but B has experienced acceleration and deceleration and therefore knows that she/he has not been in an inertial frame! This solves the paradox.

Note: in Minkowski spacetime $OO_1O_2 < OO_2$.

2.12 Experimental evidence for Special Relativity

Clearly Special Relativity is consistent with Michelson & Morley's experiment and its refined versions since.

A well know test of time dilation comes from the behaviour of muons (elementary particles formed by the collision of Cosmic rays with particles in the upper atmosphere). The mean life of muons is approximately $2.2 \times 10^{-6}s$ so that if the moved at the speed of light they could only cover a distance of approximately $0.66km$. However, they reach the ground level from heights of about $10km$. To explain this, they must have a dilation factor of approximately 15. This means they would have a speed of about $0.997c$!

From the muon's point of view, they have a normal life time, however, they depth of the atmosphere is contracted by a factor of 15,

Time dilation can also be observed using accurate atomic clocks on board of airplanes which are then compared with fixed clocks.

2.13 Proper time

In order to develop relativistic dynamics one requires the analogues of

$$\underline{v} = \frac{\mathrm{d}\underline{x}}{\mathrm{d}t}, \quad \underline{a} = \frac{\mathrm{d}\underline{v}}{\mathrm{d}t}, \quad \underline{F} = \frac{\mathrm{d}\underline{p}}{\mathrm{d}t},$$

etc. The problem is that in Special Relativity, $t = x^0$ is not a scalar, so that we cannot just carry $\mathrm{d}/\mathrm{d}t$ over to Special Relativity.

The closest thing to $\mathrm{d}t$ which is a scalar is the *proper time* interval $\mathrm{d}\tau$ defined by

$$\mathrm{d}\tau^2 \equiv -\frac{\mathrm{d}s^2}{c^2} = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$

In the previous definition the minus sign is included so that $\mathrm{d}\tau$ and $\mathrm{d}t$ have the same sign! The name of proper time comes from the fact that a clock at rest with a moving particle —i.e. in the particle's rest frame where $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = 0$— has $\mathrm{d}\tau = \mathrm{d}\tau$ —i.e. it is equal to the time elapsed on the particle's clock.

We employ τ as the invariant measure of time for the particle.

2.14 4-velocity and 4-momentum

In order to express Newton's laws in Special Relativity in an invariant way, we need to express them in terms of 4-vectors.

4-velocity

The 4-velocity of a particle is defined as a unit tangent to its Worldline:

$$\bar{U} = \frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}, \quad U^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau}.$$

Remarks:

(1) From the definition of $\mathrm{d}\tau$ one finds that

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = \mathrm{d}\bar{x} \cdot \mathrm{d}\bar{x}$$

where $\mathrm{d}\bar{x} = (\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ so that

$$\bar{U} \cdot \bar{U} = -1. \tag{2.18}$$

So that 4-velocity as defined has unit length.

(2) From $\mathrm{d}\tau^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ one finds that

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = 1 - \underline{v}^2,$$

where $\underline{v}$ denotes the 3-velocity relative to the frame F and $\underline{v}^2 = \underline{v} \cdot \underline{v}$. Hence, one concludes that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-v^2}} = \gamma(v) \quad (c = 1).$$

Now, using

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma(v)v^1, \quad \text{etc}$$

one finds that

$$\bar{U} = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}y}{\mathrm{d}\tau}, \frac{\mathrm{d}z}{\mathrm{d}\tau}\right) = \gamma(v)(1, v^1, v^2, v^3),$$

or in short

$$\bar{U} = \gamma(v)(1, \underline{v}). \tag{2.19}$$

Note that the spatial part of $\bar{U}$ is essentially $\underline{v}$.

4-momentum

The 4-momentum is the natural analogue of the 3-momentum:

$$\bar{p} = m_0\bar{U},$$

where m_0 denotes the mass of the particle. From the definition it follows that

$$\bar{p} \cdot \bar{p} = m_0^2\bar{U} \cdot \bar{U} = -m_0^2,$$

where it has been used that $\bar{U}\cdot\bar{U}=-1$. Also, using (2.19) one has

$$\bar{p}=m_0\gamma(v)(1,\underline{v}). \tag{2.20}$$

It follows that the space part of (2.20) can be identified with the 3-momentum, where by analogy $m_0\gamma$ is called the *the moving mass*, or the apparent mass and m_0 is referred as the *rest mass*.

Let

$$m\equiv m_0\gamma(v)=\frac{m_0}{\sqrt{1-v^2/c^2}},$$

so that the time component of $\bar{p}$ is identified with the energy

$$E=m_0c^2\gamma(v).$$

One reason for this identification comes from considering the limit for small v/c. For $v/c\ll 1$ one has

$$\begin{aligned}E=m_0c^2\gamma(v)&=m_0c^2(1-v^2/c^2)^{-1/2}\\&\approx m_0c^2+\tfrac{1}{2}m_0v^2,\end{aligned}$$

where the binomial expansion has been used. Now, the second term is just the Newtonian kinetic energy $(\frac{1}{2}m_0v^2)$. The first term (m_0c^2) is then interpreted as the rest mass energy. This is the famous equation

$$E_{rest}=m_0c^2.$$

From the previous discussion one can write

$$\bar{p}=(E,\underline{p}), \tag{2.21}$$

with $\underline{p}$ the 3-momentum and E the energy. From (2.20) one concludes that

$$\bar{p}\cdot\bar{p}=(E,\underline{p})\cdot(E,\underline{p})=-E^2+\underline{p}\cdot\underline{p}.$$

Using (2.18) one concludes

$$E^2-\underline{p}\cdot\underline{p}=m_0^2,\quad (c=1).$$

2.15 Photons

The definition of 4-velocity given in the previous sections breaks down when applied to particles moving with the speed of light (photons) since for light rays one has $\mathrm{d}s^2=-\mathrm{d}\tau^2=0$. In this case one may choose another parameter λ and define

$$\bar{k}=\frac{\mathrm{d}\bar{x}}{\mathrm{d}\lambda},$$

but again $\bar{k}\cdot\bar{k}=0$ since $\bar{k}$ is null. This also implies that $\bar{p}\cdot\bar{p}=0$ for photons as $\bar{p}$ is in the direction of $\bar{U}$. Now, recalling that $\bar{p}\cdot\bar{p}=-m_0^2$, it follows that $m_0=0$ for photons. Hence, particles moving wit the speed of light must be massless!

Consider a photon with 4-momentum $\bar{p}=(E,\underline{p})$ defined relative to some frame F. As seen before $\bar{p}\cdot\bar{p}=0$, so that one finds that

$$E^2-p^2=0,\qquad \text{or } E=p.$$

Therefore, for photons the spatial 3-momentum and the energy are equal. In particular, if the photon moves along the x-direction one has that

$$p_x=E.$$

2.16 Doppler shift

Let F and F' be in standard configuration. Consider a photon of frequency ν moving in the x-direction relative to the frame F. Relative to the frame F' the energy of the photon may be obtained using a Lorentz transformation. For this recall that $\bar{p}$ is a 4-vector and its energy is given by its t-component. So, from

$$\bar{p} = (E, p_x), \quad p_y = p_z = 0,$$

one obtains

$$E' = \gamma(E - vp_x), \quad (c = 1). \tag{2.22}$$

Also, recall that form Quantum Mechanics, a photon of frequency ν has energy given by $h\nu$ where h denotes Planck's constant:

$$h = 6.625 \times 10^{-34} Js.$$

Similarly, one has $E' = h\nu'$. Substituting in (2.22) one obtains

$$h\nu' = \frac{h\nu - vp_x}{\sqrt{1 - v^2}}. \tag{2.23}$$

Furthermore, for such a photon $E = p_x$ so that substituting into (2.23):

$$h\nu' = \frac{h\nu - vh\nu}{\sqrt{1 - v^2}},$$

from where

$$\frac{\nu'}{\nu} = \frac{1 - v}{\sqrt{1 - v^2}} = \sqrt{\frac{1 - v}{1 + v}}.$$

Adding the constant c:

$$\frac{\nu'}{\nu} = \sqrt{\frac{1 - v/c}{1 + v/c}}. \tag{2.24}$$

This is the *relativistic Doppler shift formula.* Note that when $v/c \ll 1$, then using the binomial expansion in (2.24) one obtains

$$\frac{\nu'}{\nu} \approx 1 - v/c,$$

which is the usual (non-relativistic) formula for the Doppler shift.

Remark. The Doppler shift has been fundamental in Cosmology to establish the expansion of the Universe.

2.17 Newton's second law and 4-acceleration

In analogy with Newton's second law, one needs to consider the rate of change of $\bar{p}$. For this, use the proper time τ as invariant candidate for time. For a particle moving with velocity $\underline{v}$ relative to F one has that

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau} = \frac{\mathrm{d}(m_0\bar{U})}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[m_0\gamma(v)(1, \underline{v})\right] = m_0 \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma(v)(1, \underline{v})\right]. \tag{2.25}$$

But,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma(v),$$

as seen in section 2.14. Also,

$$\begin{aligned}\frac{\mathrm{d}\gamma(\underline{v})}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(1-v^2)^{-1/2} = \frac{\mathrm{d}}{\mathrm{d}t}(1-\underline{v}\cdot\underline{v})^{-1/2} \\ &= -\frac{1}{2}\left(\frac{-2\underline{v}\cdot\frac{\mathrm{d}\underline{v}}{\mathrm{d}t}}{(1-\underline{v}\cdot\underline{v})^{3/2}}\right),\end{aligned}$$

so that

$$\frac{\mathrm{d}\gamma(v)}{\mathrm{d}t} = \gamma^3\underline{v}\cdot\dot{\underline{v}},$$

where we have written

$$\dot{\underline{v}} \equiv \frac{\mathrm{d}\underline{v}}{\mathrm{d}t}.$$

Substituting into (2.25):

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau} = m_0\gamma\left[\gamma(0,\dot{\underline{v}}) + \gamma^3\underline{v}\cdot\dot{\underline{v}}(1,\underline{v})\right],$$

and finally

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau} = m_0\gamma^4\left(\underline{v}\cdot\dot{\underline{v}},(1-v^2)\dot{\underline{v}} + (\underline{v}\cdot\dot{\underline{v}})\underline{v}\right).$$

Now, for $v \ll c$ one has that $\gamma \approx 1$ and $(\underline{v}\cdot\dot{\underline{v}})\underline{v} \approx \dot{v}v^2/c^2 \ll 1$ so that

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau} \approx m_0(\underline{v}\cdot\dot{\underline{v}},\dot{\underline{v}}).$$

The second term (spatial part) on the right hand side of the last equation is the usual rate of change of the 3-momentum while the time part is the rate of change of the kinetic energy.

4-acceleration

For $|\underline{v}| \ll c$ the 4-acceleration is defined as

$$\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau} \approx (\underline{v}\cdot\dot{\underline{v}},\dot{\underline{v}}).$$

with the spatial part being approximately the 3-acceleration at low $\underline{v}$. From

$$\bar{U}\cdot\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau} = 0$$

it follows that the 4-acceleration is orthogonal to the 4-velocity. Using the definition of 4-acceleration Newton's second law becomes

$$\bar{F} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau},$$

where $\bar{F}$ denotes the 4-force vector. Note also, that $\bar{F}\cdot\bar{U} = 0$ so that also $\bar{F}$ and $\bar{U}$ are orthogonal. This can be seen as follows:

$$\bar{F}\cdot\bar{U} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau}\cdot\bar{U} = m_0\frac{\mathrm{d}\bar{U}}{\mathrm{d}\tau}\cdot\bar{U} = 0.$$

2.18 3-velocity and 3-acceleration

Let F and F' be in standard configuration and moving with velocity V along the x-axis. For simplicity, we will restrict our attention to movements along the x-axis. Let v be the (uniform) velocity of a particle relative to F To find v', the velocity relative to F' recall that:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{2.26a}$$

$$v' = \frac{\mathrm{d}x'}{\mathrm{d}t'}, \tag{2.26b}$$

where the increment represents the distances and times between two events for the particle relative to the two frames. Using the inverse Lorentz transformations

$$\mathrm{d}x = \gamma(\mathrm{d}x' + V\mathrm{d}t'), \quad \mathrm{d}t = \gamma(\mathrm{d}t' + V\mathrm{d}x'),$$

in (2.26b) one obtains

$$v = \frac{\gamma(\mathrm{d}x' + V\mathrm{d}t')}{\gamma(\mathrm{d}t' + V\mathrm{d}x)} = \frac{v' + V}{1 + v'V}.$$

In the sequel we will need a transformation rule for the 3-acceleration. Starting from

$$v = \frac{v' + V}{1 + v'V}$$

and calculating the differential

$$\mathrm{d}v = \frac{\mathrm{d}v'}{1 + v'V} - \frac{v' + V}{(1 + v'V)^2} V\mathrm{d}v',$$

one concludes that

$$\mathrm{d}v = \frac{1}{\gamma^2}\frac{\mathrm{d}v'}{(1 + v'V)^2}. \tag{2.27}$$

Also, from the inverse Lorentz transformation

$$t = \gamma(t' + Vx'),$$

it follows that

$$\mathrm{d}t = \gamma(\mathrm{d}t' + V\mathrm{d}x'),$$

and furthermore that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{\gamma^3(1 + v'V)^3}\frac{\mathrm{d}v'}{\mathrm{d}t'}.$$

Notice that as a consequence of this formula, is the acceleration is zero in one inertial frame, then it is zero in all inertial frames. Hence, acceleration is in a certain sense absolute.

2.19 Uniform acceleration

In Newtonian Physics, uniform acceleration is defined as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \text{constant}.$$

It follows that

$$\lim_{t\to\infty} v = \infty,$$

which contradicts Special Relativity! Thus, in Special Relativity a definition of uniform acceleration is adopted which does not suffer from this shortcoming. One defines 3-acceleration as uniform if at each time t the acceleration of the particle relative to an inertial frame with the same velocity as the particle has the same value —i.e. if it has the same value in a comoving frame (a frame that is momentarily at rest). An example of this would be a spacecraft with engine running at constant rate.

Let a constantly accelerating particle have velocity $v = v(t)$ relative to a frame F, along its x-axis. Then, at any time t the velocity of the comoving frame F' (in which the particle is stationary) is v. Therefore the velocity of the particle relative to F' is $v' = 0$ and the 3-acceleration $\mathrm{d}v'/\mathrm{d}t'$ is a constant —say a.

Using the transformation rule for the acceleration deduced in the previous section, namely,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{\gamma^3(1+v'v)^3}\frac{\mathrm{d}v'}{\mathrm{d}t'},$$

with $v' = 0$ and $\mathrm{d}v'/\mathrm{d}t' = a_0$ one obtains

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{a_0}{\gamma} = \left(1-v^2\right)^{3/2} a_0.$$

Integrating:

$$\int_0^v \frac{\mathrm{d}v}{\left(1-v^2\right)^{3/2}} = \int_{t_0}^t a_0 \mathrm{d}t \quad v_0 = 0 \text{ at } t = t_0,$$

one obtains

$$\frac{v}{\left(1-v^2\right)^{1/2}} = a_0(t-t_0).$$

Solving for v one finds

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{a_0(t-t_0)}{\left(1+a_0^2(t-t_0)^2\right)^{1/2}}.$$

Integrating once more

$$x - x_0 = \frac{1}{a_0}\left(1+a_0^2(t-t_0)^2\right)^{1/2} - \frac{1}{a_0},$$

which can be rewritten as

$$\frac{(x-x_0+1/a_0)^2}{(1/a_0)^2} - \frac{(t-t_0)^2}{1/a_0} = 1. \tag{2.28}$$

The latter is an hyperbola in the (x,t) plane. For simplicity take $t_0 = 0$ and $x_0 = 1/a_0$ so that (2.28) reduces to

$$\frac{x^2}{(1/a_0)^2} - \frac{(t)^2}{(1/a_0)} = 1.$$

This formula gives different hyperbolae for different values of a_0.

2.20 Relativistic dynamics

In Special Relativity Newton's laws become:

First law. Remains unchanged, except that straight lines the straight lines referred to are now world lines in Minkowski spacetime.

Second law. One has

$$\bar{F} = \frac{\mathrm{d}\bar{p}}{\mathrm{d}\tau}.$$

Third law. On basis of very precise experiments of Particle Physics, this remains unchanged. That is, 4-momentum is conserved in collisions:

$$\sum_i \bar{p}_i = \text{constant},$$

where the sum is over the particles involved in the collision.

Note. Due to constancy of the time component, the conservation of energy with rest mass is included in the balance!

2.21 Examples of relativistic collisions

This type of problems can be solved by equating components, squaring and then using further properties of $\bar{p}$.

Example 1

Consider 2 particles with rest masses m_1 and m_2 both moving along collinearly with speeds u_1 and u_2. The particles collide and coalesce with the resulting particles moving in the same direction. The question is: what are the mass m and the speed u of the resulting particle?

Recall that $\bar{p} = m\gamma(1, \underline{v})$ for a particle of 3-velocity $\underline{v}$. The initial 4-momenta are:

$$\bar{p}_1 = m_1\gamma(u_1)(1, u_1, 0, 0),$$
$$\bar{p}_2 = m_2\gamma(u_2)(1, u_2, 0, 0).$$

The final 4-momentum is

$$\bar{p} = m\gamma(u)(1, u, 0, 0).$$

The conservation of -momentum is expressed by

$$\bar{p} = \bar{p}_1 + \bar{p}_2. \tag{2.29}$$

Squaring

$$\bar{p}^2 = \bar{p} \cdot \bar{p} = \bar{p}_1^2 + \bar{p}_2^2 + 2\bar{p}_1 \cdot \bar{p}_2. \tag{2.30}$$

However,

$$|\bar{p}_1|^2 = -m_1^2, \quad |\bar{p}_2|^2 = -m_2^2,$$
$$\bar{p}_1 \cdot \bar{p}_2 = m_1 m_2 \gamma(u_1)\gamma(u_2)(-1 + u_1 u_2).$$

Substituting in (2.30):

$$m = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma(u_1)\gamma(u_2)(1 - u_1 u_2)}. \tag{2.31}$$

Taking space and t-components of 4-momenta in equation (2.29)

$$m\gamma(u)u = m_1\gamma(u_1)u_1 + m_2\gamma(u_2)u_2, \tag{2.32a}$$
$$m\gamma(u) = m_1\gamma(u_1) + m_2\gamma(u_2). \tag{2.32b}$$

Dividing (2.32a) by (2.32b) one obtains

$$u = \frac{m_1\gamma(u_1)u_1 + m_2\gamma(u_2)u_2}{m_1\gamma(u_1) + m_2\gamma(u_2)}. \tag{2.33}$$

Remark. In the limit of $u_1 \ll c$ and $u_2 \ll c$ one has that $\gamma(u_1),\ \gamma(u_2) \approx 1$ and that $(1 - u_1 u_2) \approx 1$ so that (2.31) and (2.33) yield

$$m \approx m_1 + m_2,$$
$$u \approx \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2},$$

which are the classical version of the result.

Example 2

Consider the collision (scattering) of a photon of frequency ν moving in the x-direction by an electron of mass m_e in a frame in which m_e is initially at rest. Assume that the subsequent motion remains in the xy plane.

Before the collision the 4-momenta of the photon and electron are given, respectively, by

$$\bar{p}_{p_1} = (h\nu, h\nu, 0, 0),$$
$$\bar{p}_{e_1} = m_e\gamma(0)(1, 0, 0, 0), \quad \gamma(0) = 1.$$

After the collision we have that

$$\bar{p}_{p_2} = (h\nu', h\nu' \cos\alpha, h\nu' \sin\alpha, 0),$$
$$\bar{p}_{e_2} = m_e\gamma(v)(1, v\cos\beta, v\sin\beta, 0),$$

where ν' is the new photon frequency and α, β are as given in the figure.

The conservation of 4-momentum gives:

$$\bar{p}_{p_1} + \bar{p}_{e_1} = \bar{p}_{p_2} + \bar{p}_{e_2}.$$

Squaring:

$$(\bar{p}_{p_1} + \bar{p}_{e_1} - \bar{p}_{p_2}) \cdot (\bar{p}_{p_1} + \bar{p}_{e_1} - \bar{p}_{p_2}) = \bar{p}_{e_2} \cdot \bar{p}_{e_2}. \tag{2.34}$$

But,

$$\bar{p}_{e_1}^2 = \bar{p}_{e_2}^2 = -m_e^2, \quad \bar{p}_{p_1} = \bar{p}_{p_2} = 0.$$

Substituting in (2.34) one obtains

$$\bar{p}_{e_1} \cdot \bar{p}_{p_1} - \bar{p}_{e_1} \cdot \bar{p}_{p_2} = \bar{p}_{p_1} \cdot \bar{p}_{p_2},$$

from where

$$-m_e h\nu + m_e h\nu' = h^2 \nu\nu'(\cos\alpha - 1),$$

and

$$\sin^2 \alpha/2 = \frac{m_e c^2}{2h}\left(\frac{1}{\nu'} - \frac{1}{\nu}\right). \tag{2.35}$$

Similarly, to find β rewrite (2.34) as

$$(\bar{p}_{p_1} + \bar{p}_{e_2} - \bar{p}_{p_2}) \cdot (\bar{p}_{p_1} + \bar{p}_{e_2} - \bar{p}_{p_2}) = \bar{p}_{e_1} \cdot \bar{p}_{e_1}.$$

This example shows that the photon is deflected (or scattered) by and angle given by (2.35)

Chapter 3

Prelude to General Relativity

3.1 General remarks

At the time of the development of Special Relativity, physical interactions were supposed to be either gravitational or electromagnetic. Electromagnetism was already compatible with Special Relativity —i.e. invariant under Lorentz transformations. On the other hand, Newton's laws were not.

After the development of Special Relativity, what was needed was to construct a relativistic theory of gravity compatible with Special Relativity. The first attempts to construct such theory involved generalisations of Newton's laws of gravity. For example, Nordström developed a theory which was Lorentz invariant but which is incompatible with the observations —it does not produce light bending.

Einstein in 1915 succeeded in constructing a theory which is both Lorentz invariant and which s compatible with predictions. This theory is called *General Relativity.* In order to develop General Relativity, we will require some ingredients of *tensor calculus.* To understand why this mathematical tool is required, we take first a look at some of the principles that underlie the theory.

3.2 The Equivalence Principle

The Equivalence Principle amounts to the following two statements:

(1) The (equation of) motion of a (spherically symmetric) test particle (one whose own gravitational field may be neglected) in a gravitational field is independent of its mass and composition. The first verification of this statement is claimed to be Galileo's Pisa bell tower experiment —although this particular experiment probably never took place. More recent experiments like the one by Roll, Krotkov and Dicke (1964) have allowed to establish the equality to 1 part in 10^{11}.

(2) Matter (as well as every form of energy) is acted on by (an is itself a source of) gravitational field. In other words, gravity couples everything.

An immediate consequence of (2) is that it is not possible to eliminate the force of gravity in the same way that other forces may be eliminated, by for example, disconnecting power sources or by means of shielding as in the case of Faraday cages. The only other forces that behave in this way are the so-called *fictitious forces* (i.e. the centrifugal and Coriolis forces) which arise when non-inertial frames of reference are employed. The important point about these forces is that like gravity, they are proportional to the mass of the particle. This led Einstein to suspect that these and the gravitational forces should enter the theory in the same way.

To get a better feeling for this, recall that the only way one can eliminate the force of gravity is by choosing a freely falling frame —i.e. a comoving frame with the freely falling particle. This is can be visualised in the thought experiment (Gedankenexperiment) — sometimes referred to as the *lift experiment*.

The experiment suggests that there are no local experiments which distinguish non-rotating free fall in gravitational field from a uniform motion in a space free from gravitational fields. By local, here its is understood that the experiment is performed in a small region such that the variation of the gravitational field is negligible (observationally). This is another way of expressing the Equivalence Principle (all particles fall in the same way). In this sense, Special Relativity is regained locally, in the sense that the laws of Physics in a freely falling frame are compatible with Special Relativity. Alternatively, one can say that spacetime is locally Minkowskian. Furthermore, for a global theory in the presence of gravitation (i.e. GR), the geometry of spacetime must be such that it is locally Minkowskian. The natural tool to express and implement these ideas is the so-called *tensor calculus*.

3.3 Summary

In presence of gravitational fields there exist, in small regions (locally), preferred inertial frames (i.e. the non-rotating free falling frames) in which the special relativistic results hold. On a large scale, on the other hand, there are no such preferred frames, and hence one needs to treat all large scale reference frames on the same footing. This suggests that the laws of nature should be formulated in such a way that they are invariant under arbitrary transformations of coordinates (i.e. reference frames), and not just the Lorentz transformations as was the case of Special relativity.

Interpreted physically, this is called the *General Principle of Relativity* as opposed to the Special Principle of Relativity according to which laws of nature have the same form in inertial frames.

Interpreted mathematically, it is called the principle of *General Covariance* —the equations of Physics should have tensorial form.

Chapter 4

Differential Geometry and tensor calculus

In describing spacetime we wish our equations to be valid for any coordinates. Tensorial equations satisfy this property —hence their significance.

4.1 Manifolds and coordinates

Roughly speaking, a *manifold* is locally equivalent to a subset of n-dimensional Euclidean space $\mathbb{R}^n$ —i.e. made of pieces that look like open sets in $\mathbb{R}^n$ and such that the pieces may be glued together smoothly. This definition allows the notion of curved space to be made precise.

Example. The surface of S^2 (2-sphere) which is locally $\mathbb{R}^2$, even though non-locally it is curved and closed.

We view an n-dimensional manifold (also called spacetime in Relativity where $n = 4$) as a set of points each possessing a set of n coordinates $(x^0, x^1, \dots, x^{n-1})$ where each coordinate ranges over a subset of the reals or the whole reals.

Note. The first coordinate is chosen to be x^0 consistent with the notation in Special Relativity where $x^0 = t$.

An important feature of general manifolds is that we cannot assume that the whole manifold can be covered with a single (non-degenerate) coordinate system as it is the case in *Euclidean or Minkowski space.*

Example. On the surface of the sphere S^2, there are no coordinates which cover the whole surface without degeneracy —i.e. with all images being well defined.

We shall have occasion to deal with space of dimension $n \geq 2$, but cases $n = 2,\ 3,\ 4$ are of most interest.

Curves

A curve is defined as the set of points given by

$$x^i = f^i(u), \quad i = 0,\ 1,\ \dots n-1,$$

with u a parameter.

Subspaces

A subspace is defined as the set of points given by

$$x^i = f^i(u^1, u^2, \dots, u^m), \quad i = 0,\ 1,\ \dots, n-1,$$

with $m < n$. We speak of a subspace of dimension $m < n$.

We shall call a space of dimension $n-1$ a *hypersurface* because like a surface in 3-dimensional space it divides the n-dimensional space into 2 disjoint sets. This can be seen as follows: one can eliminate the $n-1$ parameters for the n equations $x^i = f^i$ leaving

$$F(x^0, x^1, \dots, x^{n-1}) = 0.$$

The points in the space not satisfying this equation fall into 2 classes —that for $F > 0$ and that for $F < 0$.

4.2 Transformation of coordinates

Assume that well behaved coordinates exist —at least in patches. Since we wish our equations to be valid for any coordinates, one has to analyse the changes from the coordinates (x^a) to (x'^a). That is, changes from

$$x'^a = x'^a(x^b) \equiv x'^a(x^0, \dots, x^{n-1}) \tag{4.1}$$

or inverse transformations of the type

$$x^a = x^a(x'^b), \tag{4.2}$$

where x^a and x'^a refer to coordinates of a point p relative to coordinates systems F and F' which are no longer assume to be inertial. We shall also assume that the functions x^a and x'^a are differentiable.

Differentiating (4.1) one obtains

$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^0}\mathrm{d}x^0 + \frac{\partial x'^a}{\partial x^1}\mathrm{d}x^1 + \cdots + \frac{\partial x'^a}{\partial x^{n-1}}\mathrm{d}x^{n-1},$$

or in a more compact form

$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^b}\mathrm{d}x^b \tag{4.3}$$

where $\partial x'^a/\partial x^b$ is the *Jacobian* of the transformation. For example, in 2 dimensions we have

$$\frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \frac{\partial x'^1}{\partial x^2} \\ \frac{\partial x'^2}{\partial x^1} & \frac{\partial x'^2}{\partial x^2} \end{pmatrix}.$$

Now, $\mathrm{d}x^a$ may be treated as an infinitesimal displacement between two neighbouring points $p(x^a)$ and $q(x^a + \mathrm{d}x^a)$.

Example

We may describe the plane $\mathbb{R}^2$ by Cartesian coordinates $(x^i) = (x, y)$ or polar coordinates $(x'^a) = (r, \theta)$. We then have

$$x'^1 = r = (x^2 + y^2)^{1/2},$$
$$x'^2 = \theta = \arctan(x/y).$$

The inverse transformations are given by

$$x^1 = x = r\cos\theta = x'^1 \cos x'^2,$$
$$x^2 = y = r\sin\theta = x'^1 \sin x'^2.$$

4.3 Contravariant vectors

The infinitesimal displacement $\mathrm{d}x^a$ is the prototype of a class of geometrical objects called *contravariant vectors*. A contravariant vector is defined as a set of n quantities V^a associated with a point p of the manifold which under change of coordinates transform according to

$$V'^a = \frac{\partial x'^a}{\partial x^b} V^b, \tag{4.4}$$

that is, in the same way as differentials. Similarly, a contravariant tensor of rank or order 2 is defined as a set of n^2 quantities U^{ab} which under a change of coordinates transform like

$$U'^{ab} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d} U^{cd}. \tag{4.5}$$

In general, a contravariant tensor of rank k is as set of quantities $V^{a_1 a_2 \cdots a_k}$ which transform according to:

$$V'^{a_1 a_2 \cdots a_k} = \frac{\partial x'^{a_1}}{\partial x^{b_1}}\frac{\partial x'^{a_2}}{\partial x^{b_2}} \cdots \frac{\partial x'^{a_k}}{\partial x^{b_k}} V^{b_1 b_2 \cdots b_k}.$$

Remark. Contravariant tensors of rank k are also called tensors of type $(k, 0)$ —e.g. a contravariant vector V^a is referred to as a tensor of rank $(1, 0)$. An important special case is a tensor of rank 0 (type $(0, 0)$) also called a scalar or an invariant:

$$\phi' = \phi \quad \text{at } p.$$

4.4 Covariant and mixed tensors

Recall that given a real valued function (scalar field) ϕ on the manifold one can define the gradient of ϕ by

$$\frac{\partial \phi}{\partial x^a}$$

which is also a vector. If we transform this expression to another coordinate system $\{x'^a\}$ we have:

$$\frac{\partial \phi}{\partial x'^a} = \frac{\partial \phi}{\partial x^b}\frac{\partial x^b}{\partial x'^a}$$

where the chain rule has been used. The latter is the prototype of a covariant vector or covariant tensor of rank 1 or of type $(0, 1)$.

More precisely, a covariant vector is defined as a set of n quantities Y_b which transform according to:

$$Y'_a = \frac{\partial x^b}{\partial x'^a} Y_b. \tag{4.6}$$

Similarly, a covariant tensor of rank 2 (or type $(0,2)$) can be defined by:

$$Y'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} Y_{cd}.$$

More generally, a covariant tensor of rank k (or type $(0,k)$) is defined as:

$$Y'_{a_1 a_2 \cdots a_k} = \frac{\partial x^{b_1}}{\partial x'^{a_1}}\frac{\partial x^{b_2}}{\partial x'^{a_2}} \cdots \frac{\partial x^{b_k}}{\partial x'^{a_k}} Y_{b_1 b_2 \cdots b_k}.$$

Important remark! It is a convention to write contravariant tensors with raised indices and covariant tensors with lowered indices.

Mixed tensors

One can also define geometric objects called *mixed tensors*. For example, the mixed tensor of rank 3 with 1 contravariant and 2 covariant indices (of type $(1,2)$) satisfies

$$Z'^a{}_{bc} = \frac{\partial x'^a}{\partial x^e}\frac{\partial x^f}{\partial x'^b}\frac{\partial x^g}{\partial x'^c} Z^e{}_{fg}.$$

Finally, one may define a mixed tensor of rank $(p+q)$ of type (p,q) —i.e. p contravariant and q covariant indices. It can be written as

$$Z^{a_1 \cdots a_p}{}_{b_1 \cdots b_q}.$$

An example

If a contravariant vector and a covariant vector have, respectively, components $A^i = (A^1, A^2)$ and $A_i = (A_1, A_2)$ in Cartesian coordinates, find the components in polar coordinates. In this example one has

$$(x'^1, x'^2) = (r, \theta),$$
$$(x^1, x^2) = (x, y).$$

Also

$$x = r\cos\theta, \quad y = r\sin\theta,$$
$$r = (x^2+y^2)^{1/2}, \quad \theta = \arctan(y/x).$$

Recall also that

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j, \quad A'_i = \frac{\partial x^j}{\partial x'^i} A_j.$$

Thus, one has to compute

$$\frac{\partial x'^i}{\partial x^j}, \quad \frac{\partial x^j}{\partial x'^i}.$$

A lengthy but straightforward calculation gives:

$$\frac{\partial x^1}{\partial x'^1} = \frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x^2}{\partial x'^1} = \frac{\partial y}{\partial r} = \sin\theta,$$
$$\frac{\partial x^1}{\partial x'^2} = \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial x^2}{\partial x'^2} = \frac{\partial y}{\partial \theta} = r\cos\theta,$$

and

$$\frac{\partial x'^1}{\partial x^1} = \frac{\partial x}{\partial x} = \frac{x}{r}, \quad \frac{\partial x'^1}{\partial x^2} = \frac{\partial r}{\partial y} = \frac{y}{r},$$
$$\frac{\partial x'^2}{\partial x^1} = \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \quad \frac{\partial x'^2}{\partial x^2} = \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

For the contravariant tensor A^a one has then that

$$A'^1 = \frac{\partial x'^1}{\partial x^a}A^a = \frac{\partial r}{\partial x}A^1 + \frac{\partial r}{\partial y}A^2 = \frac{1}{r}(xA^1 + yA^2),$$
$$A'^2 = \frac{\partial x'^2}{\partial x^a}A^a = \frac{\partial \theta}{\partial x}A^1 + \frac{\partial \theta}{\partial y}A^2 = \frac{1}{r^2}(-yA^1 + xA^2).$$

For the covariant tensor A_a one has

$$A'_1 = \frac{\partial x^a}{\partial x'^1}A_a = \frac{\partial x}{\partial r}A_1 + \frac{\partial y}{\partial r}A_2 = A_1\cos\theta + A_2\sin\theta,$$
$$A'_2 = \frac{\partial x^a}{\partial x'^2}A_a = \frac{\partial x}{\partial \theta}A_1 + \frac{\partial y}{\partial \theta}A_2 = -r\sin\theta A_1 + r\cos\theta A_2.$$

This example shows, in particular, that contravariant and covariant tensors are different geometric objects.

Remark 1. In Cartesian coordinates there exists no distinction between covariant and contravariant vectors and that is why one could get away with thinking that they are the same. In general, specially in spaces in which no global Cartesian coordinates exist, it is important to recognize that even though $\partial\phi/\partial x^a$ is a vector, it is not the same kind of vector as $\mathrm{d}x^a$.

Remark 2. Tensors as we have defined them are a set of components at a point of the manifold with particular transformation rules.

Remark 3. A tensor field is an association of a tensor of the same rank to every point of a manifold. A tensor field is called continuous or differentiable if its components in some coordinate system are continuous or differentiable functions of the coordinates. If they are C^∞, they are called smooth.

Remark 4. A vector is a tensor with one index. A scalar is a tensor with no indices. Not all geometric objects are tensors.

Remark 5. The importance of tensors in Mathematical Physics and Relativity lie in the fact that a tensor equation which holds in one coordinate system holds in all coordinate systems. For example, suppose that in unprimed coordinates one has that

$$V_{ab} = W_{ab}.$$

The transformation to primed coordinate is given by

$$
V'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}V_{cd},
$$
$$
W'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}W_{cd},
$$

so that

$$
V'_{ab} = W'_{ab}.
$$

Remark 6. The Kronecker delta $\delta^i{}_j$ is defined by

$$
\delta^i{}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
$$

Now, to prove that $\delta^i{}_j$ is a tensor of type $(1,1)$ we note that if were one it should transform as:

$$
\delta'^i{}_j = \frac{\partial x'^i}{\partial x^a}\frac{\partial x^b}{\partial x'^j}\delta^a{}_b.
$$

Now, substituting in the right hand side for $\delta^a{}_b$

$$
\frac{\partial x'^i}{\partial x^b}\frac{\partial x^b}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta'^i{}_j.
$$

4.5 Tensor algebra

To write equations in a covariant (tensorial) form, we need to build new tensors from given ones. The following simple algebraic operations are useful. The trick is always to show that a given object transforms like a tensor.

4.5.1 Addition (linear combination)

The linear combination of tensors of the same type is a tensor of the same type. As a first example we show that if $W^a{}_b$ and $Z^a{}_b$ are tensors of type $(1,1)$, then so is

$$
V^a{}_b \equiv cW^a{}_b + dZ^a{}_b,
$$

with c, d scalars. To see this notice that

$$
\begin{aligned}
V'^a{}_b &= cW'^a{}_b + dZ'^a{}_b \\
&= c\frac{\partial x'^a}{\partial x^e}\frac{\partial x^f}{\partial'^b}W^e{}_f + d\frac{\partial x'^a}{\partial x^e}\frac{\partial x^f}{\partial'^b}Z^e{}_f, \\
&= \frac{\partial x'^a}{\partial x^e}\frac{\partial x^f}{\partial'^b}V^e{}_f,
\end{aligned}
$$

which transforms as a tensor of type $(1,1)$.

4.5.2 Direct product

The product of 2 tensors of type (p_1, q_1) and (p_2, q_2) is a tensor of type $(p_1 + p_2, q_1 + q_2)$ provided none of the indices are the same. As an example, if $V^a{}_b$ and W^c are tensors of type $(1,1)$ and $(1,0)$ respectively, show that

$$Z^a{}_b{}^c \equiv V^a{}_b W^c,$$

is a tensor of type $(2,1)$. To see this

$$\begin{aligned} Z'^a{}_b{}^c &= V'^a{}_b W'^c, \\ &= \frac{\partial x'^a}{\partial x^e}\frac{\partial x^f}{\partial x'^b} V^e{}_f \frac{\partial x'^c}{\partial x^h} W^h, \\ &= \frac{\partial x'^a}{\partial x^e}\frac{\partial x^f}{\partial x'^b}\frac{\partial x'^c}{\partial x^h} Z^e{}_f{}^h. \end{aligned}$$

4.5.3 Contraction

Setting an upper and a a lower index equal and summing over its values results in a new tensor with the two indices absent. That is, one passes from a tensor of rank (p,q) to one of rank $(p-1, q-1)$. For example if $Z^a{}_b{}^{cd}$ is a tensor of type $(3,1)$, show that $Z^{ac} \equiv Z^a{}_b{}^{cb}$ is a tensor of type $(2,0)$. To see this write

$$Z'^{ac} = Z'^a{}_b{}^{cb} = \frac{\partial x'^a}{\partial x^e}\frac{\partial x^f}{\partial x'^b}\frac{\partial x'^c}{\partial x^g}\frac{\partial x'^b}{\partial x^h} Z^e{}_f{}^{gh}.$$

However, note that

$$\frac{\partial x^f}{\partial x'^b}\frac{\partial x'^b}{\partial x^h} = \frac{\partial x^f}{\partial x^h} = \delta^f{}_h,$$

so that

$$Z'^{ac} = \frac{\partial x'^a}{\partial x^e}\frac{\partial x'^c}{\partial x^g} Z^{eg}.$$

The latter is a tensor of rank $(2,0)$.

4.5.4 Detection of tensors

Suppose one has a geometric object with indices. How does one decide if it is a tensor? As an example, if B^i is an arbitrary contravariant vector and $A_i B^i$ is an invariant scalar, prove that A_i is a covariant vector. To see this write

$$\begin{aligned} A_i B^i &= A'_i B'^i \\ &= A'_i \left(\frac{\partial x'^i}{\partial x^j} B^j \right) \end{aligned}$$

so that

$$\left(A_j - A'_i \frac{\partial x'^i}{\partial x^j} \right) B^j = 0,$$

and since B^j is arbitrary this implies

$$A_j = A'_i \frac{\partial x'^i}{\partial x^j},$$

so that A_j is indeed contravariant.

4.5.5 Symmetric and antisymmetric tensors

A tensor A_{ij} is said to be *symmetric* if

$$A_{ij} = A_{ji},$$

and *antisymmetric* (or skew) if

$$A_{ij} = -A_{ji}.$$

Remark 1. For tensor this property is preserved under coordinate transformations because

$$A_{ij} \pm A_{ji}$$

are tensors by the addition property, so if they vanish in one coordinate system then they vanish in all coordinate systems.

Remark 2. In n dimensions a symmetric tensor A_{ij} has $\frac{1}{2}n(n+1)$ independent components and an antisymmetric has $\frac{1}{2}n(n-1)$ independent components.

Remark 3. Any rank 2 tensor can be expressed as the sum of a symmetric and an antisymmetric (skew) parts:

$$\begin{aligned} A_{ij} &= \tfrac{1}{2}(A_{ij} + A_{ji}) + \tfrac{1}{2}(A_{ij} - A_{ji}), \\ &= A_{(ij)} + A_{[ij]}. \end{aligned}$$

For a tensor of higher rank one says that it is symmetric (or skew) with respect to a pair of indices if interchanging the indices does not change the components (changes the sign). The indices involved must be both "upstairs" or "downstairs". For example

$$R_{[ab][cd]}$$

implies

$$\begin{aligned} R_{abcd} &= -R_{bacd}, \\ R_{abcd} &= -R_{abdc}, \\ R_{abcd} &= R_{badc}. \end{aligned}$$

One also defines

$$\begin{aligned} A_{(ijk)} &= \tfrac{1}{6}(A_{ijk} + A_{jki} + A_{kij} + A_{kij} + A_{ikj} + A_{jik}), \\ A_{[ijk]} &= \tfrac{1}{6}(A_{ijk} + A_{jki} + A_{kij} - A_{kij} - A_{ikj} - A_{jik}). \end{aligned}$$

4.6 Derivatives and connections

Most dynamical laws of Physics are expressible as differential equations. A coordinate independent formulation of such laws requires a coordinate independent definition of derivative.

It is recalled that the partial derivative of a scalar function is a tensor, however, as it will be seen the partial derivative of a higher rank tensor is not tensorial. To see this, consider a contravariant tensor V^a. Its transformation law is given by

$$V'^b = \frac{\partial x'^b}{\partial x^a} V^a.$$

Differentiating with respect to x'^c one obtains

$$\frac{\partial V'^b}{\partial x'^c} = \frac{\partial^2 x'^b}{\partial x^d \partial x^a}\frac{\partial x^d}{\partial x'^c} V^a + \frac{\partial V^a}{\partial x^d}\frac{\partial x^d}{\partial x'^c}\frac{\partial x'^b}{\partial x^a},$$

where the chain rule

$$\frac{\partial}{\partial x'^c} = \frac{\partial x^d}{\partial x'^c}\frac{\partial}{\partial x^d}$$

has been used. The second term in the right hand side is what one would expect if $\partial V^a/\partial x^d$ were a tensor of second rank. It is the first term the one that destroys the tensorial character!

One needs a definition of derivative which renders tensors. That is, a modification ∇_a of ∂_a with produces tensors. If ∇_c is to be a derivative one needs the following to be satisfied:

$$\begin{aligned}
&\nabla_c f = \partial_c f, \\
&\nabla_c (A_b + B_b) = \nabla_c A_b + \nabla_c B_b, \quad \text{(linearity)} \\
&\nabla_c (A_a B_b) = (\nabla_c A_a) B_b + A_a (\nabla_c B_b), \quad \text{(Leibnitz rule)}.
\end{aligned}$$

The simplest modification of ∂_c that satisfies the above requirements is the following:

$$\nabla_c V^a \equiv \partial_c V^a + \Gamma^a{}_{bc} V^b, \tag{4.7}$$

where the quantity $\Gamma^a{}_{bc}$ which has N^3 components is called the *connection* or sometimes the *affine connection*. Note that its particular form has not yet identified.

Notation. Very often we shall write equation (4.7) and similar expressions as

$$V^a{}_{;c} = V^a{}_{,a} + \Gamma^a{}_{bc} V^b,$$

where we have introduced the *colon-semicolon* notation:

$$V^a{}_{;c} \equiv \nabla_c V^a, \quad \partial_c V^a \equiv V^a{}_{,a}.$$

Also notice that the differentiation index c comes last in the connection $\Gamma^a{}_{bc}$.

Tensorial character of the covariant differentiation

We will now choose a transformation law for $\Gamma^a{}_{bc}$ making $\nabla_c V^a$ a tensor of type $(1,1)$. Recall that we already have seen that from

$$V'^b = \frac{\partial x'^b}{\partial x^a} V^a,$$

it follows that

$$V'^a{}_{,c} = \frac{\partial^2 x'^b}{\partial x^d \partial x^a}\frac{\partial x^d}{\partial x'^c} V^a + \frac{\partial V^a}{\partial x^d}\frac{\partial x^d}{\partial x'^c}\frac{\partial x'^b}{\partial x^a}.$$

Now, from the definition (4.7) one has

$$V'^a{}_{;c} = V'^a{}_{,c} + \Gamma'^a{}_{bc} V'^b,$$

so that

$$V'^a{}_{;c} = \frac{\partial V^b}{\partial x^d}\frac{\partial x^d}{\partial x'^c}\frac{\partial x'^a}{\partial x^b} + \frac{\partial^2 x'^a}{\partial x^d \partial x^b}\frac{\partial x^b}{\partial x'^c} V^b + \Gamma'^a{}_{bc}\frac{\partial x'^b}{\partial x^f} V^f. \tag{4.8}$$

Now, to ensure that $V'^a{}_{;c}$ transforms as a tensor of type $(1,1)$ one requires

$$\Gamma'^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\Gamma^d{}_{ef} - \frac{\partial^2 x'^a}{\partial x^d \partial x^l}\frac{\partial x^d}{\partial x'^c}\frac{\partial x^l}{\partial x'^b}.$$

The second term in this last expression is to cancel the second term in (4.8) by noting that

$$\frac{\partial x^l}{\partial x'^b}\frac{\partial x'^b}{\partial x^f} = \delta^l{}_f$$

and that f and b are dummy indices that can be interchanged:

$$-\left(\frac{\partial^2 x'^a}{\partial x^d \partial x^f}\frac{\partial x^d}{\partial x'^c}\frac{\partial x^l}{\partial x'^b}\right)\frac{\partial x'^b}{\partial x^f}V^f = -\frac{\partial^2 x'^a}{\partial x^d \partial x^f}\frac{\partial x^d}{\partial x'^c}V^f.$$

Remark 1. Clearly, $\Gamma^a{}_{bc}$ is not a tensor. Its transformation law is not homogeneous.

Remark 2. By insisting that the covariant derivative of a scalar is the partial derivative and that the covariant differentiation satisfies the Leibnitz rule, one can obtain a formula for the covariant derivative of a covariant tensor:

$$V_{a;b} = V_{a,b} - \Gamma^c{}_{ab}V_c.$$

In general, one has that

$$T^{a\cdots}{}_{b\cdots;c} = T^{a\cdots}{}_{b\cdots,c} + \Gamma^a{}_{dc}T^{d\cdots}{}_{b\cdots} + \cdots - \Gamma^d{}_{bc}T^{a\cdots}{}_{d\cdots}.$$

4.7 Parallel transport

A tensor V^a is said to be parallely transported along W^b if

$$W^b\nabla_b V^a = W^b V^a{}_{;b} = 0.$$

Now, recall that one way of characterising straight lines in Euclidean space is as curves whose tangent vectors are parallely transported at every point —i.e. they are autoparallels. The notion of shortest distance in this context is not appropriate as we have not defined a distance on the manifold —-this will be seen in the sequel.

The notion defined above can be used to define the analogue of *straight lines* in more general manifolds. Such curves are referred to as *affine geodesics* —i.e. curves along which the tangent vector is propagated parallely to itself.

Letting W^b to be tangent to a geodesic, one has that

$$W^b\nabla_b W^a = W^b W^a{}_{;b} = 0,$$

from where

$$W^b W^a{}_{,b} + \Gamma^a{}_{cb}W^c W^b = 0.$$

If the curve is parametrised by λ, then

$$W^b = \frac{\mathrm{d}x^b}{\mathrm{d}\lambda},$$

and since

$$W^b \frac{\partial}{\partial x^b} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\equiv \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \frac{\partial}{\partial x^b} \right),$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \right) + \Gamma^a{}_{bc} \frac{\mathrm{d}x^c}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0,$$

and finally that

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\lambda} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^c}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0. \tag{4.9}$$

Note. From the existence and uniqueness theorems for ordinary differential equations, it follows that corresponding to every direction at a point, there exists a unique geodesic passing through the point. The initial conditions are

$$\lambda = \lambda_0, \quad x_0^a = x^a(0), \quad W_0^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}(0).$$

Example. Show that changing the geodesic parameter λ to σ in such a way that $\sigma = \sigma(\lambda)$, the geodesic equation only keeps its form (4.9) in σ if $\sigma = a\lambda + b$.

To see this recall that

$$\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^a}{\mathrm{d}\sigma} \frac{\mathrm{d}\sigma}{\mathrm{d}\lambda},$$

so that

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}^2 x^a}{\mathrm{d}\sigma^2} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda} \right)^2 + \frac{\mathrm{d}x^a}{\mathrm{d}\sigma} \frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2}.$$

Substituting into equation (4.9) one gets

$$\left(\frac{\mathrm{d}^2 x^a}{\mathrm{d}\sigma} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^c}{\mathrm{d}\sigma} \frac{\mathrm{d}x^b}{\mathrm{d}\sigma} \right) \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda} \right)^2 + \frac{\mathrm{d}x^a}{\mathrm{d}\sigma} \frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2} = 0,$$

which only has the form of (4.9) if

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\lambda^2} = 0.$$

That is, if

$$\sigma = a\lambda + b.$$

A parameter of this form is called an *affine parameter*.

Remark. Note that only the symmetric part of the connection coefficient is required in the geodesic equation (4.9).

4.8 Manifolds with metric

So far, in addition to tensor fields, our manifold had a connection defined on it. This allows for the notions of differentiation and parallelism.

There are some reasons that lead us to introduce further structure on the manifold. Namely,

(i) from the Equivalence principle, spacetime is locally Minkowskian;

(ii) the need of an alternative notion of parallelism based on the idea of length;

(iii) finding a relation between covariant and contravariant tensors.

In order to accomplish these point we introduce the notion of *metric*. This essentially amounts to defining the distance between two neighbouring points x^a and $x^a + \mathrm{d}x^a$ through an expression of the form

$$\mathrm{d}s^2 = g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b \tag{4.10}$$

where $\mathrm{d}s^2$ is called the *line element* or *interval* and g_{ab} is the *metric tensor*. A metric with such a metric defined on it is called a *manifold with metric*.

Remark 1. The tensor g_{ab} is a tensor of type $(0, 2)$. This follows immediately from the scalar nature of $\mathrm{d}s^2$ and the fact that $\mathrm{d}x^a$ is a contravariant tensor —this from the tensor detection tensor. To see this recall that

$$\mathrm{d}x^a = \frac{\partial x^a}{\partial x'^c}\mathrm{d}x'^c,$$

so that

$$\begin{aligned}
\mathrm{d}s^2 &= g_{ab}\mathrm{d}x^a\mathrm{d}x^b \\
&= g_{ab}\frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}\mathrm{d}x'^c\mathrm{d}x'^d \\
&= g'_{cd}\mathrm{d}x'^c\mathrm{d}x'^d,
\end{aligned}$$

where

$$g'_{cd} = g_{ab}\frac{\partial x^a}{\partial x'^c}\frac{\partial x^b}{\partial x'^d}.$$

The later is precisely the transformation law for a tensor $(0, 2)$.

Remark 2. In order for (4.10) to determine g_{ab} uniquely, g_{ab} must be symmetric. Note that if g_{ab} is symmetric, then it can always be diagonalised. Let λ_i, $i = 0, \ldots N$ denote the eigenvalues of g_{ab}. If all the eigenvalues of g_{ab} are positive, then the metric g_{ab} will be said to be a *Riemannian metric* and the manifold will be said to be a *Riemannian* manifold. If one of the eigenvalues is negative and the remaining positive, then the metric will be said to be *Lorentzian* —this is the case of relevance in Relativity. The number of positive eigenvalues minus the number of negative eigenvaues is calld the *signature*. For example, the Minkowski metric of Special Relativity (see below) has signature 2, while the standard Euclidean metric in $\mathbb{R}^4$ has signature 4.

Remark 3. Euclidean space with

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2$$

is an example of a Riemannian manifold. On the other hand, Minkowski space with

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

is a special case of Lorentzian manifold. Note that in both examples, the coefficients g_{ab} are constants. We also note that the Minkowski metric can be written is spherical coordinates as:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2.$$

The definition (4.10) allows the following natural definitions of notions one had in Minkowski space.

Norm of a covariant vector V^a

This is defined via

$$|V|^2 \equiv g_{ab}V^aV^b.$$

If $|V|^2 > 0$ (or $|V| < 0$) for all vectors V^a, the metric is called positive definite (or negative definite) —this is the Riemannian case. Otherwise it is called indefinite —this includes the Lorentzian case.

Scalar product between two vector A^a and B^b

This is defined via

$$A \cdot B \equiv g_{ab}A^aB^b.$$

If $g_{ab}A^aA^b = 0$, then A^a and B^b are said to be orthogonal.

Null vectors

For indefinite metrics there are vectors that are orthogonal to themselves. That is,

$$g_{ab}A^aA^b = 0.$$

Contravariant form of the metric

Let $g \equiv \det(g_{ab})$. If $g \neq 0$, then the inverse of g_{ab}, g^{ab} can be defined by

$$g_{ab}g^{ac} = \delta_b{}^c.$$

Defined in this way, g^{ab} is a contravariant tensor of rank 2.

As an example, consider the case when g_{ab} is diagonal —that is, $g_{ab} = 0$ $a \neq b$. Then one can show that for $g \neq 0$,

$$g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g^{22}}, \quad \cdots, g^{ab} = 0, \ \ a \neq b.$$

Lowering and raising of indices (index gymnastics)

One can use g_{ab} and g^{ab} to lower and raise indices for general tensors via the rules:

$$\begin{aligned} T^{\cdots}_{\cdots a} &\equiv g_{ab}T^{\cdots b}_{\cdots} \quad (a \text{ is the raised index}) \\ T^{\cdots a}_{\cdots} &\equiv g^{ab}T^{\cdots}_{\cdots b} \quad (a \text{ is the raised index}). \end{aligned}$$

For example

$$\begin{aligned} g_{ac}W^{ab} &= W_c{}^b, \\ T^{ab} &= g^{ac}T_c{}^b = g^{ac}g^{bd}T_{cd}. \end{aligned}$$

Quite crucially, one can see that the operation of lowering and raising indices does not add extra information in the tensors. For example, given

$$V^b = g^{ba}V_a,$$

one has that

$$g_{eb}V^b = g_{eb}g^{ba}V_a = \delta_e{}^aV_a = V_e.$$

Thus, if one raises an index and then one lowers it, one receovers the orginal tensor.

Connection between contravariant and covariant vectors

So far, covariant and contravariant tensors have remained unrelated. The metric can be taken as the mapping between contravariant and covariant tensors:

$$V_a = g_{ab}V^b, \quad V^a = g^{ab}V_b.$$

For Euclidean space in Cartesian coordinates

$$g_{ab} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

This is the reason why in this case there is no distinction between covariant and contravariant tensors.

Remark 1. Raising and lowering of indices enable us to write equations with indices in any position. It is for that reason that one writes a blank space above each lower index and below each upper index. For example the contravariant version of

$$G_{ab} = T_{ab}$$

is

$$G^{ab} = T^{ab}.$$

Remark 2. In a general 4-dimensional Lorentzian manifold with a metric g_{ab} with $g \neq 0$ the following theorem holds:

Theorem 1. *Given any point p of a Lorentzian manifold, it is always possible to find coordinate transformations with origin at p such that*

$$g_{ab}(x) = \eta_{ab} + O(x^2).$$

That is, g_{ab} is approximately the Minkowski metric to second order:

$$g_{ab}(p) = \eta_{ab}, \quad g_{ab,c}(p) = 0, \quad g_{ab,cd}(p) \neq 0.$$

One also has

Theorem 2. *In the diagonal form of g_{ab} the number of components equal to $+1$ and to -1 do not change under coordinate transformations. The difference between these two is called the signature of the metric.*

For example, the signature of the Minkowski metric is $+2$.

4.9 The Levi-Civita connection

Up to now the connection $\Gamma^a{}_{bc}$ has remained undefined. We will see now that given a metric g_{ab}, there is a preferred (canonical) connection. For this we will impose two conditions on the connection.

No torsion condition

Let ϕ be as scalar. The usual partial derivatives acting on a scalar commute. That is,

$$\partial_a \partial_b \phi = \partial_b \partial_a \phi.$$

This is, in general, not the case for covariant derivatives. Recall that

$$\nabla_b \phi = \partial_b \phi,$$

so that

$$\nabla_a \nabla_b \phi = \partial_a \partial_b \phi - \Gamma^e{}_{ba} \partial_e \phi,$$
$$\nabla_b \nabla_a \phi = \partial_b \partial_a \phi - \Gamma^e{}_{ab} \partial_e \phi.$$

Thus, the covarinat derivatives commute if and only if

$$\Gamma^c{}_{ab} = \Gamma^c{}_{ba} = \Gamma^c_{(ab)}.$$

This property has a nice geometric interpretation. namely, that the parallelogram formed by the parallel propagation of two infinitesimal displacements closes.

Constancy of the inner product upon parallel propagation

The metric g_{ab} imposes a natural condition on the parallel transport. Given two vectors V^a and W^b one can require that their inner product $g_{ab} V^a W^a$ remains unchanged if we parallel transport them along any curve. Thus, we require

$$T^a \nabla_a (g_{bc} V^a W^b) = 0,$$

with V^a and W^b satisfying

$$T^a \nabla_a V^b = 0, \quad T^a \nabla_a W^b = 0.$$

Using the Leibnitz rule one obtains

$$T^a V^b W^c \nabla_a g_{bc} = 0.$$

This equation should hold for all curves and parallely transported vectors if and only if

$$\nabla_a g_{cd} = g_{cd;a} = 0.$$

Theorem 3. *Let $g_{ab} = 0$ be a metric. Then there exists a unique connection such that*

$$\nabla_a g_{bc} = 0.$$

To prove this start from

$$0 = \nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d{}_{ba} g_{dc} - \Gamma^d{}_{ca} g_{bd},$$

so that

$$\Gamma_{cab} + \Gamma_{bac} = \partial_a g_{bc},$$

where $\Gamma_{cab} \equiv g_{dc} \Gamma^d{}_{ab}$. By index substitution one also has that

$$\Gamma_{cba} + \Gamma_{abc} = \partial_b g_{ac},$$
$$\Gamma_{bca} + \Gamma_{acb} = \partial_c g_{ab}.$$

Adding the first two equations, subtracting the third and using the symmetry ${\Gamma^c}_{ab} = {\Gamma^c}_{ba}$ one finds

$$2\Gamma_{cab} = \partial_a g_{bc} + \nabla_b g_{ac} - \nabla_c g_{ab}.$$

That is,

$${\Gamma^c}_{ab} = \tfrac{1}{2} g^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right).$$

This is called the *Levi-Civita* connection of the metric g_{ab}.

4.10 Metric geodesics

In Euclidean geometry, straight lines are defined as the shortest distance between any two points. Here we give an analogue of this for a manifold with a metric.

Recall that in Lorentzian manifolds, straight lines are not those with shortest distances (intervals) between 2 points, but the longest. The generalisation of a straight line —a geodesic line— turns out to be the curve of extremal path (i.e. maximal or minimal). In order to find extrema, one needs some elements of *calculus of variations*. Let

$$L = L(x, \dot{x}, \lambda), \quad x = x(\lambda), \quad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\lambda}.$$

That is, L is a function of functions of λ —L is called a functional. It is assumed that L is differentiable in x, $\dot{x}$, λ.

We are looking for the necessary conditions on the function x such that the integral

$$\int_{x_1}^{x_2} L(x, \dot{x}, \lambda) \mathrm{d}\lambda$$

is stationary (i.e. a maximum or a minimum) with respect to changes in the function x.

The required condition is called the *Euler-Lagrange equation* and takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0. \tag{4.11}$$

This expression can be generalised to the case where L is a function of N independent functions, $x^i(\lambda)$, $i = 1, \ldots, N$, provided they can be varied independently. In that case (4.11) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0, \tag{4.12}$$

corresponding to N equations, one for each value of i.

To deduce the geodesic equation we want to consider the length of the curve defined by $\int \mathrm{d}s$ to be stationary. Introducing a parameter λ along the curve such that

$$\int \mathrm{d}s = \int \frac{\mathrm{d}s}{\mathrm{d}\lambda} \mathrm{d}\lambda,$$

the problem becomes that of finding the extremals of

$$L = \frac{\mathrm{d}s}{\mathrm{d}\lambda} = \sqrt{g_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda}} = \sqrt{g_{jk} \dot{x}^i \dot{x}^j}.$$

Alternatively, one can find extremals of

$$L = \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2 = g_{jk}\dot{x}^i\dot{x}^j.$$

A computation renders

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}^c} &= g_{ab}\frac{\partial \dot{x}^a}{\partial \dot{x}^c}\dot{x}^b + g_{ab}\dot{x}^a\frac{\partial \dot{x}^b}{\partial \dot{x}^c}, \\ &= g_{ab}\delta^a{}_c\dot{x}^b + g_{ab}\dot{x}^a\delta^b{}_c, \\ &= g_{cb}\dot{x}^b + g_{ac}\dot{x}^a = 2g_{ac}\dot{x}^a.\end{aligned}$$

Now, recall that the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^e}{\mathrm{d}\lambda}\frac{\partial}{\partial x^e} = \dot{x}^e\partial_e.$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) &= 2\frac{\mathrm{d}g_{ac}}{\mathrm{d}\lambda}\dot{x}^a + 2g_{ac}\frac{\mathrm{d}\dot{x}^a}{\mathrm{d}\lambda} \\ &= 2\partial_e g_{ac}\dot{x}^e\dot{x}^a + 2g_{ac}\ddot{x}^a.\end{aligned}$$

Finally,

$$\frac{\partial L}{\partial x^c} = \partial_c g_{ab}\dot{x}^a\dot{x}^b.$$

Thus, one has that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 2g_{ac}\ddot{x}^a + (\partial_b g_{ac} + \partial_a g_{bc} - \partial_c g_{ab})\dot{x}^a\dot{x}^b.$$

Multiplying by $\frac{1}{2}g^{fc}$ one obtains

$$\ddot{x}^f + \Gamma^f{}_{ab}\dot{x}^a\dot{x}^b = 0,$$

which can be rewritten as

$$\frac{\mathrm{d}^2x^f}{\mathrm{d}\lambda^2} + \Gamma^f_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = 0. \tag{4.13}$$

This is the geodesic equation which we have met already. Thus, "straight lines" are also extremal.

Remark 1. In Euclidean space in Cartesian coordinates or in Minkowski space in Minkowski coordinates all the Christoffel symbols vanishes and equation (4.13) becomes

$$\frac{\mathrm{d}^2x^l}{\mathrm{d}s^2} = 0,$$

which is the usual equation for straight motion.

Remark 2. As it stands, the above equation only makes sense for spacelike curves for which $\mathrm{d}s^2 > 0$. For timelike curves one uses $\mathrm{d}\tau$ instead. Also, starting with $\int \mathrm{d}s^2$ gives the same geodesic equation.

Remark 3. For null geodesics, i.e. geodesics for which $\mathrm{d}s = 0$, the curve may be parametrised by a parameter

$$\frac{\mathrm{d}^2 x^l}{\mathrm{d}u^2} + \Gamma^l{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}u}\frac{\mathrm{d}x^k}{\mathrm{d}u} = 0,$$

where

$$g_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}u}\frac{\mathrm{d}x^k}{\mathrm{d}u} = 0.$$

Remark 4. It can be proved that if g_{ab} is Riemannian then the solutions to equation (4.13) are curves of minimum length. On the other hand, if g_{ab} is Lorentzian, then the geodesics maximise length. Now, recall that in Special Relativity one defines the proper time as $\mathrm{d}\tau^2 = -\mathrm{d}s^2/c^2$. Thus, time observed by a comoving clock always goes slower.

4.11 Calculation of Christoffel symbols and geodesic equations using the metric

There are 2 ways to do this: either directly using the definition of the Christoffel symbols or by using the geodesic equation.

As an example consider the 2-dimensional metric

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \cos^2 u \mathrm{d}v^2.$$

4.11.1 Computation using the definition of the Christoffel symbols

Recall that

$$\Gamma^l{}_{ij} = \tfrac{1}{2} g^{lk}(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}).$$

For our metric one has that

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\cos^2 u \end{pmatrix}.$$

Now, for a 2 dimensional space, the Christoffel symbols have $2^3 = 8$ components:

$$\begin{aligned}
&\Gamma^1{}_{11} = \tfrac{1}{2} g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = 0 \quad \text{since } g_{11,1} = 0 \\
&\Gamma^1{}_{12} = \tfrac{1}{2} g^{11}(g_{11,2} + g_{12,1} - g_{12,1}) = 0 \quad \text{since } g12 = 0 \text{ and } g_{11,2} = 0 \\
&\Gamma^1{}_{21} = 0, \\
&\Gamma^1{}_{22} = \tfrac{1}{2} g^{11}(g_{12,2} + g_{12,2} - g_{22,1}) = -\tfrac{1}{2} g^{11} g_{22,1} = \sin u \cos u, \\
&\Gamma^2{}_{12} = \tfrac{1}{2} g^{22}(g_{21,2} + g_{22,1} - g_{12,2}) = \tfrac{1}{2} g^{22} g_{22,1} = -\tan u, \\
&\Gamma^2{}_{21} = -\tan u, \\
&\Gamma^2{}_{11} = \Gamma^2{}_{22} = 0.
\end{aligned}$$

The problem of this approach is that one need to calculate all the components, one by one.

4.11.2 Computation using the Euler-Lagrange equations

This is usually a more useful way as it gives directly the non-zero Christoffel symbols. Let

$$L = \frac{\mathrm{d}s}{\mathrm{d}\lambda} = \dot{u}^2 + \cos^2 u \dot{v}^2.$$

The Euler-Lagrange equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0.$$

Look at the different components. For $i = 1$ one has

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(2\dot{u}) - (-2\sin u \cos u \dot{v}^2) = 0,$$

so that

$$\ddot{u} + \sin u \cos u \dot{v}^2 = 0. \tag{4.14}$$

The latter is equivalent to (cfr. (4.13)):

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}s^2} + \Gamma^1{}_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0,$$

or

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}s^2} + \Gamma^1{}_{11}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^1{}_{12}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^1{}_{21}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} + \Gamma^1{}_{22}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} = 0.$$

However, in our case one only has $\dot{v}^2$ terms so the latter becomes

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}s^2} + \Gamma^1{}_{22}\left(\frac{\mathrm{d}x^2}{\mathrm{d}s}\right)^2 = 0.$$

The latter in combination with gives

$$\Gamma^1{}_{22} = \sin u \cos u, \quad \Gamma^1{}_{11} = \Gamma^1{}_{12} = \Gamma^1{}_{21} = 0.$$

For $i = 2$ one finds from

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(2\dot{v}\cos^2 u) = 0,$$

so that

$$\ddot{v} - 2\dot{u}\dot{v}\tan u = 0. \tag{4.15}$$

Again, from the equation for the geodesic one has that

$$\frac{\mathrm{d}^2 x^2}{\mathrm{d}s^2} + \Gamma^2{}_{12}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^2}{\mathrm{d}s} + \Gamma^2{}_{21}\frac{\mathrm{d}x^2}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} = 0.$$

However,

$$\Gamma^2{}_{12} = \Gamma^2_{21},$$

and hence

$$\Gamma^2{}_{12} = \Gamma^2_{21} = -\tan u.$$

Finally,

$$\Gamma^2{}_{22} = \Gamma^2{}_{11} = 0.$$

Chapter 5

Curvature

A novel feature of General Relativity is that it employs the notion of *curved space*. Our intuition of curvature is mainly based on the curvature of 2-dimensional objects in 3-dimensional space, like spheres, saddles, etc. The notion of curvature whose definition depends on a space of higher dimension is called *extrinsic*. In the case of spacetime this notion is not useful and require an *intrinsic* notion —i.e. a definition which is independent of the embedding space.

5.1 Intrinsic curvature and the Riemann tensor

Gauss showed that for a general 2-dimensional surface with a metric of the form

$$\mathrm{d}s^2 = g_{11}(x^1, x^2)(\mathrm{d}x^1)^2 + g_{22}(x^1, x^2)(\mathrm{d}x^2)^2,$$

where it has been assumed that $g_{12} = 0$ for simplicity, it is possible to define an intrinsic curvature (a scalar function) which is invariant under coordinate transformations, but varies from point to point. This is given by an expression of the form

$$K = F[g_{ij}, \partial_k g_{ij}, \partial_l \partial_k g_{ij}],$$

such that when $K = 0$ the space is flat and for a sphere of radius R it gives $K = 1/R^2$.

In spaces of higher dimension we require more than one quantity at each point to describe curvature. It turns out that the right definition involves the components of a 4-index tensor called the *Riemann curvature tensor*:

$$R^a{}_{bcd} \equiv \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}. \tag{5.1}$$

Since the Christoffel symbols contain derivatives of the metric, one finds that the Riemann tensor has the same form as K. Note that in flat space given by Cartesian coordinates the Christoffel symbols vanish, and thus the Riemann tensor vanishes! If one shows that $R^a{}_{bcd}$ is indeed a tensor, then this last statement is valid for any coordinates! This statement is actually an if and only if statement. The hard part is to show that vanishing curvature implies Minkowski space.

There are many ways of motivating this formula. Here we will proceed by looking at the commutation of covariant derivatives. Consider:

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = V_{a;b;c} - V_{a;c;b}.$$

Now recall that

$$V_{a;b} = V_{a,b} - \Gamma^d{}_{ab} V_d,$$

so that

$$\begin{aligned}
V_{a;b;c} &= (V_{a;b})_{,c} - \Gamma^f{}_{ac} V_{f;b} - \Gamma^f{}_{bc} V_{a;f} \\
&= \left(V_{a,b} - \Gamma^d{}_{ab} V_d\right)_{,c} - \Gamma^f{}_{ac}\left(V_{f,b} - \Gamma^d{}_{fb} V_d\right) - \Gamma^f{}_{bc}\left(V_{a,f} - \Gamma^d{}_{af} V_d\right) \\
&= V_{a,b,c} - \partial_c \Gamma^d{}_{ab} V_d - \Gamma^d{}_{ab} V_{d,c} - \Gamma^f{}_{ac} V_{f,b} + \Gamma^f{}_{ac}\Gamma^d{}_{fb} V_d - \Gamma^f{}_{bc} V_{a,f} + \Gamma^f{}_{bc}\Gamma^d{}_{af} V_d.
\end{aligned}$$

Interchanging b and c in the last expression:

$$V_{a;c;b} = V_{a,c,b} - \partial_b \Gamma^d{}_{ac} V_d - \Gamma^d{}_{ac} V_{d,b} - \Gamma^f{}_{ab} V_{f,c} + \Gamma^f{}_{ab}\Gamma^d{}_{fc} V_d - \Gamma^f{}_{cb} V_{a,f} + \Gamma^f{}_{cb}\Gamma^d{}_{af} V_d.$$

Thus,

$$\begin{aligned}
V_{a;b;c} - V_{a;c;b} = (V_{a,b,c} - V_{a,c,b}) \\
&+ \left(\Gamma^d{}_{ac} V_{d,b} - \Gamma^f{}_{ac} V_{f,b}\right) + \left(\Gamma^f{}_{ab} V_{f,c} - \Gamma^d{}_{ab} V_{d,c}\right) \\
&+ \left(\Gamma^f{}_{bc} V_{a,f} - \Gamma^f{}_{cb} V_{a,f}\right) + \left(\Gamma^f{}_{bc}\Gamma^d{}_{af} V_d - \Gamma^f{}_{cb}\Gamma^d{}_{af} V_d\right) \\
&+ \left(\partial_b \Gamma^d{}_{ac} V_d - \partial_c \Gamma^d{}_{ab} V_d + \Gamma^f{}_{ac}\Gamma^d{}_{fb} V_d - \Gamma^f{}_{ab}\Gamma^d{}_{fc} V_d\right)
\end{aligned}$$

The first term of the right hand side cancels out as usual partial derivatives commute. The second and third cancel out directly, while in the fourth and fifth we use the symmetry of the Christoffel symbols. Thus, one is left with

$$\begin{aligned}
V_{a;b;c} - V_{a;c;b} &= V_d\left(\partial_b \Gamma^d{}_{ac} - \partial_c \Gamma^d{}_{ab} + \Gamma^f{}_{ac}\Gamma^d{}_{fb} - \Gamma^f{}_{ab}\Gamma^d{}_{fc}\right) \\
&= V_d R^d{}_{abc},
\end{aligned}$$

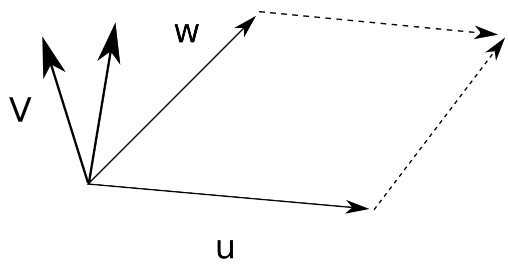
as it can be seen by comparison with equation (5.1). This expression is sometimes called the Ricci identity. Defined through this expression, if follows that the $R^d{}_{abc}$ is indeed a tensor as the expression in the left hand side is a tensor —alternatively, one could look at the transformation rules of the Christoffel symbols. This is much more involved!

Geometric interpretation

It can be shown that the change of a vector V^c parallely transported along a closed path is proportional to the curvature —see figure. For an infinitesimal loop along the directions given by u^b and w^d one has that

$$\delta V^a = R^a{}_{cbd} V^c \delta u^b \delta w^d.$$

Recall that as seen before such parallelogram closes (due to the no Torsion condition)!



5.2 Symmetries of the curvature tensor

In general, a tensor of rank 4 has $4^4 = 256$ components (in spacetime). Symmetries, if present are important because they reduce the number of independent components. Lowering the index in the definition of the Riemann tensor one obtains

$$R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc} + \Gamma_{aec}\Gamma^e{}_{bd} - \Gamma_{aed}\Gamma^e{}_{bc},$$

where

$$R_{abcd} = g_{af} R^f{}_{bcd}, \quad \Gamma_{abd} = g_{af}\Gamma^f{}_{bd}.$$

Now, since R_{abcd} is a tensor, it should have the same symmetries in all frames. Accordingly, choose a *locally inertial frame* for which the Christoffel symbols vanish. For these coordinates one has then that

$$R_{abcd} = \partial_c \Gamma_{abd} - \partial_d \Gamma_{abc}.$$

Recalling that

$$\Gamma_{abc} = \tfrac{1}{2}\left(g_{ab,c} + g_{ac,b} - g_{bc,a}\right)$$

one obtains

$$R_{abcd} = \tfrac{1}{2}\left(g_{ad,bc} + g_{bc,ad} - g_{bd,ac} - g_{ac,bd}\right),$$

from where it is easy to read the symmetries of the tensor. It can be checked that

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}, \quad R_{abcd} = R_{cdab}.$$

Furthermore,

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \quad \text{so that } R_{a(bcd)} = 0.$$

These symmetries amount to 236 constraints, so R_{abcd} has only 20 non-zero components.

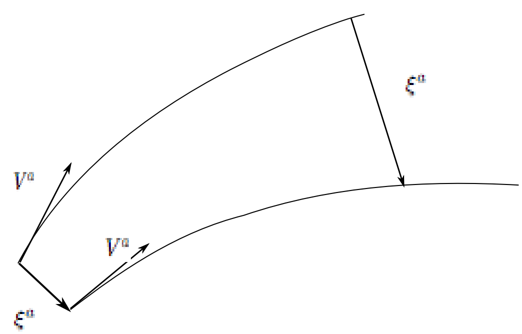
5.3 Geodesic deviation

Parallel lines in curved space do not remain parallel when extended. For this one considers two nearby geodesics with tangent given by V^a and a vector ξ^a describing its separation. The evolution of the separation vector ξ^a is described by the equation

$$\nabla_V \nabla_V \xi^a = R^a{}_{cdb} V^c V^d \xi^b,$$

which is called the *geodesic deviation equation.* Note that the curvature is non-zero then ξ^a changes! In this last equation ∇_V denotes the directional derivative with respect to V^a —that is, $\nabla_V \equiv V^a \nabla_a$.

Remark. The last equation shows that if particles follow geodesics (an assumption made in General Relativity) then the tidal gravitational forces that make the trajectories to converge can be mathematically represented by the curvature of spacetime!



5.4 Bianchi identities, the Ricci and Einstein tensors

Recall that in a locally inertial frame one had that

$$R_{abcd} = \tfrac{1}{2}(g_{bc,ad} + g_{ad,bc} - g_{bd,ac} - g_{ac,cd}).$$

Differentiating with respect to x^e one obtains

$$R_{abcd,e} = \tfrac{1}{2}(g_{bc,ade} + g_{ad,bce} - g_{bd,ace} - g_{ac,cde}).$$

Using the fact that partial derivatives commute one finds that

$$R_{abcd,e} + R_{abec,d} + R_{abde,c} = 0, \quad R_{ab(cd,e)} = 0.$$

Now, in a locally inertial frame the Christoffel symbols vanish so that in fact one has that

$$R_{abcd;e} + R_{abec;d} + R_{abde;c} = 0, \quad R_{ab(cd;e)} = 0.$$

This tensorial equation is valid in all frames and is called the *Bianchi identity*. One could have derived it by directly taking the covariant derivative of the Riemann tensor.

The Ricci tensor

The Ricci tensor is obtained by contracting the first and third indices of the Riemann tensor:

$$R_{bd} \equiv g^{ac} R_{abcd} = R^c{}_{bcd}.$$

Remark 1. Because of the symmetries of the Riemann tensor one has that the Ricci tensor is symmetric. That is,

$$R_{bd} = R_{db}.$$

Remark 2. Other contractions of the Riemann tensor vanish or give $\pm R_{bd}$. For example $R^b{}_{bcd} = 0$ since R_{abcd} is symmetric on a and b. Also,

$$R^a{}_{bda} = -R^a{}_{bad} = -R_{bd},$$

and similarly.

The Ricci scalar

The Ricci scalar is defined as the contraction of the indices of the Ricci tensor:

$$R \equiv g^{ab} R_{ab} = g^{ac} g^{bd} R_{abcd}.$$

The Einstein tensor

In the next computations recall that $g_{ab;c} = 0$ and $g^{ab}{}_{;c} = 0$. Consider the Bianchi identity, contract with g^{ac} and bringing g^{ac} into the covariant derivative:

$$(g^{ac} R_{abcd})_{;e} + (g^{ac} R_{abec})_{;d} + (g^{ac} R_{abde})_{;c} = 0. \tag{5.2}$$

Now,

$$g^{ac} R_{abcd} = R_{bd}, \quad g^{ac} R_{abec} = -g^{ac} R_{abce},$$

so that (5.2) renders

$$R_{bd;e} - R_{be;d} + R^c{}_{bde;c} = 0.$$

Contracting on b and d:

$$R_{;e} - R_{be;d} - R^c{}_{e;c} = 0, \tag{5.3}$$

where it has been used that

$$g^{bd}R^c{}_{bde} = g^{bd}g^{ca}R_{abde} = -g^{ca}g^{bd}R_{bade} = -g^{ca}R_{ae} = -R^c{}_e.$$

On can rewrite equation (5.3) as

$$(2R^c{}_e - \delta^c{}_e R)_{;c} = 0. \tag{5.4}$$

Raising e one gets

$$(2R^{cd} - g^{cd}R)_{;c} = 0.$$

Defining

$$G^{cd} \equiv R^{cd} - \tfrac{1}{2}Rg^{cd}$$

one has

$$G^{cd}{}_{;c} = 0.$$

The tensor G^{cd} is called the *Einstein tensor*. Observe that one can also lower the indices:

$$G_{fe} = R_{fe}\tfrac{1}{2}Rg_{fe}.$$

Remark 1. The Einstein tensor is symmetric (from the symmetries of the Ricci and metric tensors) and therefore it has 10 independent components.

Remark 2. By construction, the Einstein tensor is *divergence free.*

Chapter 6

General Relativity

6.1 Towards the Einstein equations

There are several ways of motivating the Einstein equations. The most natural is perhaps through considerations involving the Equivalence Principle. In gravitational fields there exist local inertial frames in which Special Relativity is recovered. The equation of motion of a free particle in such frames is:

$$\frac{\mathrm{d}^2 x'^a}{\mathrm{d}\tau^2} = 0.$$

Relative to an arbitrary (accelerating frame) specified by $x^a = x^a(x'^b)$, the latter becomes:

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \gamma^a{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0,$$

where

$$\gamma^a{}_{bc} = \frac{\partial x^a}{\partial x'^d}\frac{\partial^2 x'^d}{\partial x^b \partial x^c}.$$

Here the $\gamma^a{}_{bc}$ are the "fictitious" terms that arise due to the non-inertial nature of the frame.

Now, due to the Equivalence Principle the latter implies that locally gravity is equivalent to acceleration and this in turn gives rise to non-inertial frames. The main idea of General relativity is to argue that gravitation as well as inertial forces should be described by appropriate $\gamma^a{}_{bc}$'s!

The simplest way to do this is by means of a Lorentzian manifold —the latter is endowed with geodesics of the required type:

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0.$$

Now, if the $\Gamma^{a\prime}_{bc}s$ are associated with gravitational forces, then the metric g_{ab} may be associated with a potential. Note that the gravitational potential in the Newtonian theory satisfies

$$\nabla^2\phi = 4\pi G\rho, \quad \rho \text{ the density.}$$

The relativistic analogue of this equation should be tensorial and of second order in the metric. To take this analogy further, consider two neighbouring particles with coordinates

$x^\alpha(t)$, $x^\alpha(t)+\xi^\alpha(t)$, with $\xi^\alpha(t)$ small $\alpha = 1,2,3$, moving in a gravitational field with a potential ϕ. the equations of motion are then given:

$$\ddot{x}^\alpha = -\frac{\partial\phi(x)}{\partial x^\alpha}$$

and

$$\ddot{x}^\alpha + \ddot{\xi}^\alpha = -\frac{\partial\phi(x)}{\partial x^\alpha} - \xi^\beta \frac{\partial^2\phi}{\partial x^\alpha \partial x^\beta} + O(\xi^2).$$

Subtracting the two last equations:

$$\ddot{\xi} = -\xi^\beta \frac{\partial^2\phi}{\partial x^\alpha \partial x^\beta}.$$

This is the relative acceleration of two test particles separated by by a 3-vector ξ^α —the second derivative of the potential gives the tidal forces. This is in analogy to the geodesic deviation equation:

$$\nabla_{\bar{V}}\nabla_{\bar{V}}\xi^\alpha = R^a{}_{cdb}V^cV^d\xi^b,$$

provided that one identifies

$$-\xi^\beta \frac{\partial^2\phi}{\partial x^\alpha \partial x^\beta}, \qquad \text{and } R^a{}_{cdb}V^cV^d\xi^b.$$

This identification would make clear the relation between gravity and geometry —note that the Riemann tensor involves second derivatives of the metric tensor.

The main idea underlying General Relativity is that matter (including energy) curves spacetime (assumed to be a Lorentzian manifold). This in turn affects the motion of particles and light rays, postulated to move on timelike and null geodesics of the Lorentzian manifold, respectively.

6.2 The principles employed in General Relativity

(1) Equivalence Principle.

(2) Principle of General Covariance. This states that laws of Nature should have tensorial form.

(3) Principle of minimal gravitational coupling. This is used to derive the General Relativity analogues of Special Relativity results. For this change

$$\eta_{ab} \to g_{ab}, \quad \partial \to \nabla.$$

For example in Special Relativity the equations for a perfect fluid are given by:

$$T^{ab} = (\rho + p)V^aV^b - p\eta^{ab},$$
$$T^{ab}{}_{,b} = 0.$$

In General Relativity these should be changed to:

$$T^{ab} = (\rho + p)V^aV^b - pg^{ab},$$
$$T^{ab}{}_{;b} = 0.$$

(3) Correspondence principle. General relativity must agree with Special Relativity in absence of gravitation and with Newtonian gravitational theory in the case of weak gravitational fields and in the non-relativistic limit (slow speed).

6.3 The Einstein equations in vacuum

In vacuum 9such as in the outside of a body in empty space) one has that the density ρ vanishes and the equation for the Newtonian potential becomes:

$$\nabla^2\phi = 0.$$

The Laplace equation involves an object with two indices ($\partial^2\phi/\partial x^i\partial x^j$). As a result, what one needs is an object with two indices —a contraction of the Riemann tensor, like the Ricci tensor:

$$R_{bc} = 0.$$

The latter are called the *Einstein vacuum field equations.* In fact, the most general form of the vacuum equations which is tensorial and depends linearly on second derivatives of the metric is:

$$R_{bc} = \Lambda g_{ab},$$

where Λ is the so-called *Cosmological constant.*

Remark 1. Outside Cosmology, Λ is usually taken to be zero.

Remark 2. The vacuum equations are a set of ten partial differential equations for the components of the metric tensor g_{ab}. These are hard to solve, apart from simple settings.

Remark 1. The Einstein equations are the simplest compatible with the Equivalence Principle, but they are not the only ones.

6.4 Newtonian limit

Consider a slowly moving particle in a weak stationary gravitational field. Recall the geodesic equation:

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0. \tag{6.1}$$

For a slow moving particle $\mathrm{d}x^\alpha/\mathrm{d}\tau$ ($\alpha = 1, 2, 3$) may be neglected relative to $\mathrm{d}t/\mathrm{d}\tau$, so that (6.1) implies that

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} + \Gamma^a{}_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0. \tag{6.2}$$

Since the gravitational field is assumed to be stationary, all t-derivatives of g_{ab} vanish and therefore

$$\Gamma^a{}_{00} = -\tfrac{1}{2}g^{ad}\frac{\partial g_{00}}{\partial x^d}. \tag{6.3}$$

Furthermore, since the field is weak, one may adopt a local coordinate system in which

$$g_{ab} = \eta_{ab} + h_{ab}, \quad |h_{ab}| \ll 1. \tag{6.4}$$

Substitution into (6.3) one has that

$$\Gamma^a{}_{00} = -\tfrac{1}{2}\eta^{ad}\frac{\partial h_{00}}{\partial x^d}.$$

Substituting in (6.2):

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} = \tfrac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\nabla h_{00}, \quad \nabla \equiv \eta^{\alpha\beta}\frac{\partial}{\partial x^\beta}, \tag{6.5a}$$

$$\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} = 0, \quad \text{as } h_{00,0} = 0. \tag{6.5b}$$

From (6.5a) it follows that $\mathrm{d}t/\mathrm{d}\tau$ is a constant. Also, from

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau},$$

it follows that

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2},$$

which in our case reduces to

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} = \frac{\mathrm{d}^2x^\alpha}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2.$$

Combining the latter with (6.5a)

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}t} = \tfrac{1}{2}\nabla h_{00}. \tag{6.6}$$

The corresponding Newtonian result is

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}t} = -\nabla\phi \tag{6.7}$$

where ϕ is the gravitational potential which far from a central body of mass M at a distance r is given by

$$\phi = -\frac{GM}{r}.$$

Comparing (6.6) and (6.7) one finds then that

$$h_{00} = -2\phi + \text{constant}.$$

However, at large distances from M one has that $\phi \to 0$ (gravity becomes negligible) and $g_{ab} \to \eta_{ab}$ (the space becomes flat). Therefore the constant must be zero so that

$$h_{00} = -2\phi.$$

Substituting in (6.4) on finds

$$g_{00} = -(1 + 2\phi).$$

Now, recall that ϕ has dimensions of (velocity)^2, $[\phi] = [GM/R] = L^2/T^2$. Therefore one has that ϕ/c^2 at the surface of the Earth is $\sim 10^{-9}$, one the surface of the Sun $\sim 10^{-6}$ and at the surface of a white dwarf $\sim 10^{-4}$. It follows that in most cases the distortion produced by gravity is in g_{ab} very small.

6.5 Applications of General Relativity

In general, the Einstein field equations are extremely complicated set of non-linear partial differential equations. In some simple settings, analytic solutions may be found. These include:

(i) The vacuum spherically symmetric static case (the Schwarzschild spacetime).

(ii) The weak field case (gravitational waves).

(iii) The isotropic and homogeneous case (Cosmology).

Usually assume that $\Lambda = 0$, except for Cosmology.

6.6 The Schwarzschild solution

This is the basis for nearly all the tests of General Relativity. The solution corresponds to the metric corresponding to a static, spherically symmetric gravitational field in the empty spacetime surrounding a central mass (like the Sun).

Choosing coordinates (t, r, θ, φ), it can be shown that a metric of this type is of the form:

$$\mathrm{d}s^2 = -e^{A(r)}\mathrm{d}t^2 + e^{B(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{6.8}$$

where $A(r)$ and $B(r)$ describe deviation of the metric from Minkowski spacetime. Note that for constant t and r the metric reduces to the standard metric for the surface of a sphere. As one is dealing with vacuum, one is poised to solve

$$R_{ab} = 0. \tag{6.9}$$

Substituting (6.8) in (6.9), and after some algebra, the only non-zero components of (6.9) have the form:

$$R_{rr} = R_{11} = \tfrac{1}{2}A'' - \tfrac{1}{4}A'B' + \tfrac{1}{4}A'^2 - \frac{B'}{r},, \tag{6.10a}$$

$$R_{\theta\theta} = R_{22} = e^{-B}\left(1 + \tfrac{1}{2}r(A' - B')\right) - 1,, \tag{6.10b}$$

$$R_{\varphi\varphi} = R_{33} = R_{22}\sin^2\theta, \tag{6.10c}$$

$$R_{tt} = R_{00} = -e^{A-B}\left(\tfrac{1}{2}A'' - \tfrac{1}{4}A'B' + \tfrac{1}{4}A'^2 + \frac{A'}{r}\right), \tag{6.10d}$$

where $'$ denotes differentiation with respect to r.

To solve $R_{ab} = 0$, we start by looking at the combination:

$$R_{rr} + e^{B-A}R_{tt} = -\tfrac{1}{2}(B' + A') = 0.$$

Integrating one obtains

$$A = -B.$$

One can without loss of generality change t to absorb the constant of integration. Substituting in (6.10b):

$$e^A(1 + rA') - 1 = 0.$$

The latter can be rewritten as

$$(re^A)' = 1,$$

which can be integrated to give

$$re^A = r + \sigma, \quad \sigma \text{ a constant}$$

so that

$$e^A = 1 + \frac{\sigma}{r},$$

so that the metric one obtains is given by

$$\mathrm{d}s^2 = -\left(1 + \frac{\sigma}{r}\right)\mathrm{d}t^2 + \left(1 + \frac{\sigma}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

To fix σ, recall that in the Newtonian limit of a central mass M,

$$g_{00} = -\left(1 - \frac{2GM}{r}\right).$$

Comparing with

$$-\left(1+\frac{\sigma}{r}\right),$$

one finds that

$$\sigma = -2GM.$$

Hence, at the end of the day one has

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2). \tag{6.11}$$

The latter is called the *Schwarzschild metric.*

Remark 1. This solution how the presence of mass curves flat spacetime.

Remark 2. The metric (6.11) is asymptotically flat. That is, it becomes Minkowskian as $r \to \infty$.

Remark 3. The solution only applies to the exterior of a star.

Remark 4. *The Birkhoff Theorem:* a spherically symmetric solution in vacuum is necessarily static. That is, there is no time dependence is spherically symmetric solutions.

6.7 Experimental tests of General Relativity

The classical experimental tests of General Relativity are based on the Schwarzschild solution. These are based on the comparison of the trajectories of freely falling particles and light rays in gravitational field of a central body with their counterparts in Newtonian theory.

In order to derive the geodesics in Schwarzschild spacetime, it is best to use the Euler-Lagrange equations with

$$L = \left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^2 = \left(1-\frac{2MG}{r}\right)\dot{t}^2 - \left(1-\frac{2MG}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2),$$

where $'$ denotes differentiation with respect to the parameter λ. For timelike geodesics one has that $\lambda=\tau$ so that $L=1$. On the other hand, for null geodesics $L=0$.

The Euler-Lagrange equations read then

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2A\dot{t}\right) = 0,, \tag{6.12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(2\dot{r}A^{-1}) - 2r(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2) + \dot{r}^2A^{-2}A' + \dot{t}^2A' = 0, \tag{6.12b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\dot{\varphi}^2 = 0, \tag{6.12c}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(r^2\sin^2\theta\dot{\varphi}) = 0, \tag{6.12d}$$

where

$$A(r) = \left(1-\frac{2GM}{r}\right),$$

and $'$ denotes differentiation with respect to r. It turns out that it is simpler to use

$$\left(1-\frac{2MG}{r}\right)\dot{t}^2 - \left(1-\frac{2MG}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2) = 1,\ 0. \tag{6.13}$$

This is, in fact, an integral of motion of the Euler-Lagrange equation. It expresses the fact that the square of the norm of the 4-velocity vector of a timelike particle is -1, while that of a photon is 0. This is like in Special Relativity.

As in Classical Mechanics (central force orbit), let us look for solutions in the Equatorial plane: $\theta = \pi/2$. It follows then that $\dot{\theta} = 0$, and from (6.12c) with $\cos\theta = 0$ one finds that $\ddot{\theta} = 0$. The orbits remain in a plane! This is like in Classical Mechanics —conservation of angular momentum.

Now, from (6.12d) it follows that

$$r^2\dot{\varphi} = h, \quad h \text{ a constant,} \tag{6.14a}$$

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = l, \quad l \text{ a constant,}. \tag{6.14b}$$

Substituting (6.14a) and (6.14b) in (6.13) one obtains

$$l^2\left(1 - \frac{2GM}{r}\right)^{-1} - \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 - \frac{h^2}{r^2} = 1,\ 0. \tag{6.15}$$

As in Newtonian theory, let $u = 1/r$ so that

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\lambda} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \dot{\varphi}\frac{\mathrm{d}r}{\mathrm{d}\varphi}, \quad \frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}.$$

Using (6.14a) one finds

$$\dot{r} = -h\frac{\mathrm{d}u}{\mathrm{d}\varphi}.$$

Then equation (6.15) in (u, φ) coordinates become

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 = \frac{l^2 - 1}{h^2} + \frac{2GM}{h^2}u + \frac{2GMu^3}{c^2}, \quad \text{for timelike geodesics,} \tag{6.16a}$$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 = \frac{l^2}{h^2} + \frac{2GMu^3}{c^2}, \quad \text{for null geodesics.} \tag{6.16b}$$

The speed of light c has been added for dimensional reasons. These are the analogues of energy equations in Newtonian theory. One can solve (6.17a)-(6.17b) approximately. For this, differentiate the equations with respect to φ:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{2GM}{h^2} + \frac{3GMu^2}{c^2}, \quad \text{for timelike geodesics,} \tag{6.17a}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{3GMu^2}{c^2}, \quad \text{for null geodesics.} \tag{6.17b}$$

From here, one has to analyse the two cases separately.

6.7.1 Timelike case —an orbiting particle

The appropriate equation (6.17a) is identical to the equation for Newtonian orbits except for the last term. This last term is small relative to other terms for planetary orbits. The ration of the last two terms for Mercury is $\sim 10^{-7}$. As a consequence of this, equation (6.17a) will be solved using perturbation methods. For this let

$$a \equiv \frac{GM}{h^2}, \quad \epsilon = \left(\frac{3GM}{c^2}\right)\frac{GM}{h^2},$$

where ϵ is dimensionless and assumed to be small. Then equation (6.17a) implies

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = a + \frac{\epsilon}{a}u^2. \tag{6.18}$$

We will look for solutions of the type

$$u = u_0 + \epsilon u_1 + O(\epsilon^2).$$

Substitution in (6.18) yields

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\varphi^2} + u_0 + \epsilon\left(\frac{\mathrm{d}^2 u_1}{\mathrm{d}\varphi^2} + u_1\right) = a + \frac{\epsilon}{a}u_0^2 + O(\epsilon^2). \tag{6.19}$$

Equating zeroth order terms in ϵ in equation (6.19) one obtains

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\varphi^2} + u_0 = a,$$

which can be solved to give

$$u_0 = a + b\cos\varphi, \quad b \text{ a constant}, \tag{6.20}$$

where without loss of generality we have set $\varphi_0 = 0$. Now, equating the first order term in (6.19) one has

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\varphi^2} + u_1 = \frac{u_0^2}{a}.$$

Substituting for u_0 from (6.20), and using that

$$\cos^2\varphi = \tfrac{1}{2}(1 + \cos 2\varphi),$$

one obtains

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\varphi^2} + u_1 = \left(a + \frac{b^2}{2a}\right) + 2b\cos\varphi + \frac{b^2}{2a}\cos 2\varphi, \tag{6.21}$$

which is a linear inhomogeneous ordinary differential equation. Its general solution consists of a general solution to the homogeneous part plus a particular solution corresponding to each term of the right hand side of (6.21). One has that for:

$$\begin{aligned}
&a + \frac{b^2}{2a}, && \text{the solution is } a + \frac{b^2}{2a}, \\
&2b\cos\varphi && \text{the solution is } b\varphi\sin\varphi, \\
&\frac{b^2}{2a}\cos 2\varphi && \text{the solution is } -\frac{b^2}{6a}\cos 2\varphi.
\end{aligned}$$

Hence, the solution to the orbit equation to first order in ϵ is

$$\begin{aligned}
u &= u_0 + \epsilon u_1 \\
&= \left(a + \epsilon a + \frac{b^2}{2a}\epsilon\right) + \left(b\cos\varphi - \frac{b^2}{6a}\epsilon\cos 2\varphi\right) + \epsilon b\varphi\sin\varphi.
\end{aligned}$$

It is observed that only the last term in this expression is non-periodic, and hence, any irregularity in the orbit (non-periodicity) must relate to this term. To see the effect of this term recall that

$$\cos\epsilon\varphi \approx 1, \quad \sin\epsilon\varphi \approx \epsilon\varphi,$$

so that

$$\cos(\varphi - \epsilon\varphi) = \cos\varphi\cos\epsilon\varphi + \sin\varphi\sin\epsilon\varphi = \cos\varphi + \epsilon\varphi\sin\varphi.$$

Hence, we write the solution as

$$u = a + b\cos(\varphi - \epsilon\varphi) + \epsilon\left(a + \frac{b^2}{2a} - \frac{b^2}{6a}\cos 2\varphi\right),$$

that is:

$$u = a + b\cos(\varphi - \epsilon\varphi) + (\text{periodic terms}).$$

New, recall that the perihelion of a planet around the Sun occurs when r is a minimum(u maximum). Now, $\cos(\varphi - \epsilon\varphi)$ is a maximum when

$$\varphi(1-\epsilon) = 2n\pi, \qquad \text{or approximately } \varphi \approx 2n\pi(1+\epsilon).$$

Successive perihelia occur then at intervals of

$$\Delta\varphi \approx 2\pi(1+\epsilon),$$

instead of 2π as in the case of periodic motion. Therefore, the perihelion shift per revolution ($\delta\varphi = \Delta\varphi - 2\pi$) is

$$\delta\varphi = 2\pi\epsilon = \frac{6\pi G^2 M^2}{h^2 c^2}.$$

From Newtonian theory we have that

$$\alpha = \frac{h^2}{GM(1-e^2)}, \quad T^2 = \frac{4\pi^2\alpha^3}{GM},$$

where e is the eccentricity of the orbit, α the semi-major axis, and T the period. One obtains then that

$$\delta\varphi = \frac{24\pi^3\alpha^2}{c^2T^2(1-e^2)}.$$

For the case of the planet Mercury this gives a total shift of $43.03''$ per century which is in good agreement with the classically unaccounted shift of $43.11'' \pm 0.45''$.

Remark 1. This is one of the famous classical tests of General Relativity, the so-called *perihelium shift of Mercury.*

Remark 2. The effect is largest in Mercury because of its high eccentricity and small period which results in a large shift.

Remark 3. For Venus one has a predicted shift of $8.6''$ and an observed of $8.4'' \pm 4.8''$. For the Earth one has $3.8''$ and $5.0'' \pm 1.2''$. For the asteroid Icarus $10.3''$ and $9.8'' \pm 0.8''$.

6.7.2 Null geodesics

The relevant equation for null geodesics is given by (6.17b):

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{3GMu^2}{c^2}.$$

As before, the term GMu/c^2 is small relative to u so let $\epsilon \equiv 3GM/c^2$, and rewrite (6.17b) as

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \epsilon u^2, . \tag{6.22}$$

As before, look for solutions of the form

$$u = u_0 + \epsilon u_1 + O(\epsilon^2).$$

Substituting into (6.22) one has

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\varphi^2} + u_0 + \epsilon \frac{\mathrm{d}^2 u_1}{\mathrm{d}\varphi^2} + \epsilon u_1 = \epsilon u_0^2 + O(\epsilon^2).$$

Equation the zero terms in the previous equation:

$$\frac{\mathrm{d}^2 u_0}{\mathrm{d}\varphi^2} + u_0 = 0,$$

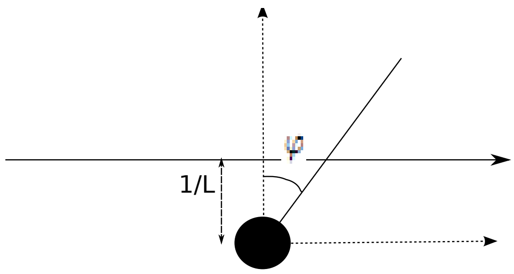
which can be solved to give

$$u_0 = L\cos\varphi, \quad L \text{ a constant.}$$

Now, $u_0 = 1/r$ so that

$$r\cos\varphi = \frac{1}{L},$$

which represents a straight line —which is what is expected. To zeroth order light is not deflected by the gravitational field of the Sun.



Equating terms of order one in ϵ one obtains

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}\varphi^2} + u_1 = u_0^2 = L^2\cos^2\varphi = \tfrac{1}{2}L^2(1 + cos2\varphi)$$

which has the particular solution

$$u_1 = \tfrac{1}{2}L^2 - \tfrac{1}{6}L^2\cos 2\varphi = \tfrac{2}{3}L^2 - \tfrac{1}{3}L^2\cos^2\varphi.$$

Hence, one obtains that

$$u = u_0 + \epsilon u_1 = L\cos\varphi + \tfrac{2}{3}L^2 - \tfrac{1}{3}L^2\cos^2\varphi. \tag{6.23}$$

So, the effects of the first order terms (the last 2 terms) is to make light deflect from a straight line.

For a light ray grazing the Sun and arriving at Earth, the asymptote of the trajectory corresponds to values of φ for which $r \to \infty$ —that is, $u \to 0$. Substituting in (**??**) this gives

$$\cos^2\varphi - \frac{3}{\epsilon L}\cos\varphi - 2 = 0.$$

The latter can be solved to give

$$\cos\varphi = \frac{3}{2L\epsilon}\left(1 \pm \sqrt{1+\tfrac{8}{9}\epsilon^2 L}\right).$$

Choosing the negative sign to make $\cos\varphi < 1$ and expanding one finds

$$|\cos\varphi| \approx \left|\tfrac{2}{3}\epsilon L\right| = \left|\frac{2}{c^2}GML\right| \ll 1,$$

which implies $\varphi \approx \pi/2$. Let $\varphi = \pi/2 + \delta$ son that

$$\sin\delta \approx 2GML/c^2 \approx \delta.$$

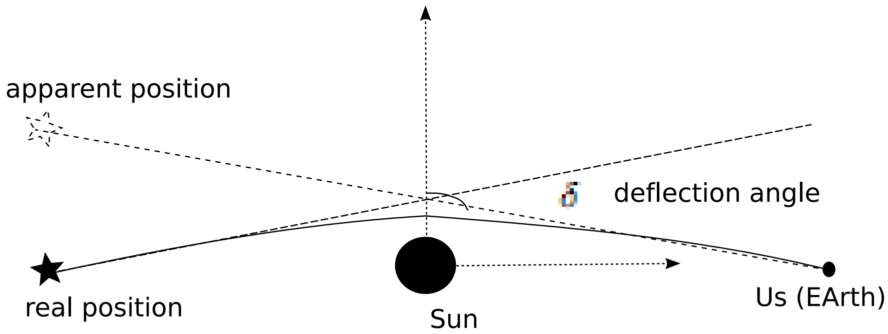
One has that δ is the angle that each asymptote makes with undeflected straight line. The angle between 2 asymptotes is

$$\Delta = 2\delta.$$

Accordingly, one finds

$$\Delta = \frac{4GML}{c^2}.$$

For a light ray just grazing the Sun this predicts a deflection of $1.75''$ which compares well with some recent radio observations yielding $\Delta = 1.73'' \pm 0.05''$.



Remark. This is the second famous test of General Relativity —more generally referred to as *bending of light.* A first measurement was carried out by Eddington and collaborators in 1919.

6.8 Gravitational redshift

Consider a clock or an atom at fixed (r, θ, φ). Hence,

$$\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\varphi = 0.$$

Then the Schwarzschild metric implies that:

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2GM}{r}\right)^{-1/2},$$

from where it follows that $\mathrm{d}t$ (i.e. the period or interval as measured by an observer at infinity) is larger than $\mathrm{d}\tau$ as measured at r. Similarly, the period of atomic oscillations in the gravitational field of M, as seen from infinity is increased. Therefore, the frequency ν ($\nu = 1/\tau$) of light it emits is decreased —i.e. *redshifted* as seen from infinity.

We define *redshift* via

$$z \equiv \frac{\lambda_{rec} - \lambda_{em}}{\lambda_{em}} = \frac{\nu_{em}}{\nu_{rec}} - 1.$$

Accordingly,

$$1 + z = \frac{\tau_{rec}}{\tau_{em}} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2GM}{r}\right)^{-1/2}.$$

Note that $z \to \infty$ as $r \to 2GM/c^2$.

Remark. It used to be thought that gravitational redshift also constituted a test of General Relativity, but it turns out that any other theory compatible with the Equivalence Principle will predict a redshift.

6.9 Black holes

Looking at the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{6.24}$$

one sees that there is something peculiar happening at

$$r = \frac{2GM}{c^2}, \quad r = 0.$$

To get a better feel of what is happening, it is best to look at coordinate independent scalars at these points. A good candidate for this is $R_{abcd}R^{abcd}$ for which the metric (6.24) is proportional to $1/r^6$. The latter is clearly singular at $r = 0$, with severe physical consequences such as that tidal forces diverge. Nevertheless the scalar remains well behaved at $r = 2GM/c^2$.

One say that at $r = 0$ one has a physical singularity wheres at $r = 2GM/c^2$ on has a coordinate singularity. In order to understand this better choose a new coordinate $\tilde{t}$ via

$$\tilde{t} = t + 2GM \ln|r - 2GM|.$$

The latter are called *Eddington-Finkelstein coordinates.* Using this new time coordinate on finds that

$$\mathrm{d}t = \mathrm{d}\tilde{t} - \frac{2GM}{r - 2GM}\mathrm{d}r,$$

so that the metric (6.24) transforms into

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}\tilde{t}^2 + \frac{4GM}{r}\mathrm{d}\tilde{t}\mathrm{d}r + \left(1 + \frac{2GM}{r}\right)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{6.25}$$

Note that in this coordinates all metric coefficients are well behaved, except for $r = 0$.

To understand the light con structure in the spacetime, we look at radial light cones defined by

$$\theta = \text{constant}, \quad \varphi = \text{constant}, \quad \mathrm{d}s^2 = 0,$$

so that

$$\left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}r}\right)^2\left(1-\frac{2GM}{r}\right)-\frac{4GM}{r}\left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}r}\right)-\left(1+\frac{2GM}{r}\right)=0.$$

Solving the quadratic equation one finds

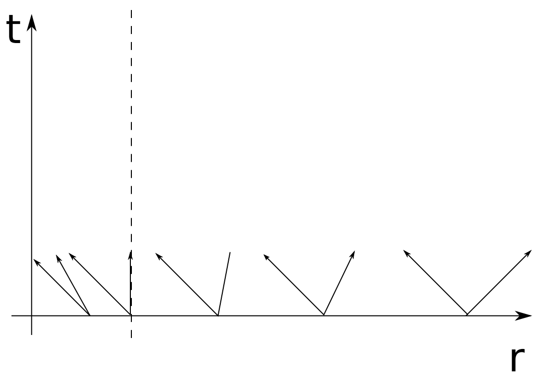
$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}r}=-1,\ \frac{r+2GM}{r-2GM}.$$

Therefore, as $r\to\infty$ one has that $\mathrm{d}\tilde{t}/\mathrm{d}r\to(-1,1)$.

For $r\to 2GM$ $\mathrm{d}\tilde{t}/\mathrm{d}r\to(-1,\infty)$.

For $r\to 0$ $\mathrm{d}\tilde{t}/\mathrm{d}r\to(-1,-1)$.

One sees that as r decreases, the outgoing path $\mathrm{d}\tilde{t}/\mathrm{d}r>0$ becomes steeper and steeper and eventually slopes inwards for $r<2GM$. Therefore, nothing, including light can go from $r<2GM$ to $r>2GM$. This justifies the name *black hole*.



Note also that even though a particle can sit stationary at $r>2GM$ as we do on Earth, given appropriate forces to counter gravity to stop our free fall), this cannot be done in a region $r<2GM$ as all timelike (null) trajectories must make an angle with the vertical, and therefore must have decreasing r for increasing $\tilde{t}$. Therefore, the particle must fall towards $r\to 0$. There is no static behaviour in the region $r<2GM$ and no escape from the singularity once inside.

Infalling particles (observers)

Another interesting question is to compare the picture as seen by an observer at infinity with that seen by an infalling observer. For this, consider radially infalling free particles into a black holes of mass M. Compare the time elapsed on the particle's clock τ for a particle to reach $r=2GM/c^2$ to time measured by a clock at rest at infinity —i.e. t, as $r\to\infty$, $\mathrm{d}\tau\to\mathrm{d}t$. Now, consider the timelike geodesics derived in section 6.7.1 and make them radial by letting $\mathrm{d}\theta=\mathrm{d}\varphi=0$. One obtains the equations

$$\left(1-\frac{2GM}{r}\right)t'=l,$$
$$\left(1-\frac{2GM}{r}\right)t'^2-\left(1-\frac{2GM}{r}\right)^{-1}r'^2=1.$$

Combining these equations one obtains

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = r'^2 = -1 + \frac{2GM}{r} + l^2.$$

However, l is arbitrary and can be chosen as $l = 1$. Hence,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2GM}{r}},$$

where the negative root gives the infalling behaviour. Integrating one finds

$$\int_{\tau_0}^{\tau} \mathrm{d}\tau = -\frac{1}{\sqrt{2GM}} \int_{r_0}^{r} \sqrt{r}\mathrm{d}r,$$

so that

$$\tau - \tau_0 = \frac{2}{3\sqrt{2GM}} \left(r_0^{3/2} - r^{3/2}\right).$$

It follows that nothing special happens to this particle at $r = 2GM/c^2$ (the Schwarzschild radius) and the body falls to $r = 0$ in a finite proper time τ. This allows to calculate the time taken for a large body (such a galaxy) to collapse.

Tom see what happens in coordinate time t, one can proceed from $\mathrm{d}t/\mathrm{d}r$ and integrate. Alternatively, one can start with the light cones of the Schwarzschild solution (radial light rays) given by

$$\mathrm{d}s^2 = 0, \quad \mathrm{d}\theta = \mathrm{d}\varphi = 0.$$

The Schwarzschild metric then implies

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{1}{1 - 2GM/r},$$

so that as $r \to \infty$, $\mathrm{d}t/\mathrm{r} \to \pm 1$, and $asr \to 2GM$, $\mathrm{d}t/\mathrm{r} \to \pm\infty$.

Therefore, the light cones close up as one approaches $r \to 2GM$. Since particles can only move within light cones, this makes them move more and more vertically. The approach takes an infinite amount of t-time!