MTH4101 CALCULUS II

REVISION NOTES

1. COMPLEX NUMBERS (Thomas Appendix 7 + lecture notes)

1.1 Introduction

Types of numbers (natural, integers, rationals, reals)

The need to solve quadratic equations:

$$ax^2 + bx + c = 0$$

Solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What happens if $b^2 < 4ac$? The need for complex numbers.

1.2 Complex Numbers

$$\mathrm{i} = \sqrt{-1}$$

$$z = x + \mathrm{i}y$$

x is the real part, y is the imaginary part.

$$x = \mathrm{Re}(z) \qquad y = \mathrm{Im}(z)$$

$$\mathrm{i}^2 = -1, \quad \mathrm{i}^3 = -\mathrm{i}, \quad \mathrm{i}^4 = 1, \quad \text{etc.}$$

$$\mathrm{i}^{-1} = -\mathrm{i}, \quad \mathrm{i}^{-2} = -1, \quad \mathrm{i}^{-3} = \mathrm{i}, \quad \text{etc.}$$

Argand diagram

Connection with polar coordinates ($x = r\cos\theta,\ y = r\sin\theta$).

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$$

$|z| = r = \sqrt{x^2 + y^2}$ is the modulus

$\arg z = \theta = \tan^{-1}(y/x)$ is the argument

Complex conjugate: if $z = x + \mathrm{i}y$ then $\bar{z} = x - \mathrm{i}y$

1.3 Operations with Complex Numbers

$$z_1 = x_1 + \mathrm{i}y_1 \qquad z_2 = x_2 + \mathrm{i}y_2$$

Addition:

$$z_1 + z_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$$

Geometric interpretation (addition of vectors in parallelogram in Argand diagram)

Subtraction:

$$z_2 - z_1 = (x_2 - x_1) + \mathrm{i}(y_2 - y_1)$$

Geometric interpretation (completion of parallelogram in Argand diagram)

Multiplication:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1)$$

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)\}$$

(i.e. multiply moduli and add arguments)

Division:

$$\frac{z_2}{z_1} = \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2} + \mathrm{i}\frac{x_1 y_2 - x_2 y_1}{x_1^2 + y_1^2}$$

$$\frac{z_2}{z_1} = \frac{r_2}{r_1}\{\cos(\theta_2 - \theta_1) + \mathrm{i}\sin(\theta_2 - \theta_1)\}$$

(i.e. divide moduli and subtract arguments)

1.4 Loci and Regions

Recognise circles (including displaced ones), lines, regions defined by inequalities; use of modulus and argument to define loci, regions.

1.5 Trigonometric Functions and Hyperbolic Functions

Recognise power series for e^z, $\sin z$, $\cos z$.

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta \qquad \text{(Euler's relation)}$$

$$z = r\,e^{\mathrm{i}(\theta+2k\pi)}$$

$$\cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2}$$

$$\sin\theta = \frac{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}{2\mathrm{i}}$$

$$e^{\mathrm{i}\pi/2} = \mathrm{i}, \quad e^{\mathrm{i}\pi} = -1, \quad e^{3\mathrm{i}\pi/2} = -\mathrm{i}, \quad e^{2\mathrm{i}\pi} = 1$$

$$\frac{\cosh x = e^x + e^{-x}}{2}$$

$$\frac{\sinh x = e^x - e^{-x}}{2}$$

Other hyperbolic functions and identities.

1.6 de Moivre's Theorem

$$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

Applications:

Expansion of $\cos^n\theta$, $\sin^n\theta$, $\cos n\theta$, $\sin n\theta$

Solution of equations (e.g. n complex roots of $z^n - 1 = 0$)

Evaluation of integrals

2. PARTIAL DERIVATIVES

2.1 Functions of Several Variables (Thomas 14.1)

Domain and range

Sketch surfaces defined by $z = f(x, y)$

Draw and label curves in the domain in which f has a constant value

Level curves

Level contours

2.2 Limits and Continuity in Higher Dimensions (Thomas 14.2)

Definition of limit for $f(x, y)$

Calculate limts of polynomials and rational function by evaluating the function at the limit point

Definition of continuous function

Two-Path Test for non-existence of a limit (if a function has different limits along two different paths then the limit does not exist); use of polar coordinates if necessary

2.3 Partial Derivatives (Thomas 14.3)

Definition of partial derivative

Notation (difference between, e.g. $\mathrm{d}/\mathrm{d}x$ and $\partial/\partial x$; meaning of f_x, f_y etc.)

Higher derivatives

Mixed Derivatives Theorem:

$$f_{xy}(a, b) = f_{yx}(a, b) \qquad \text{etc.}$$

Definition of differentiability

2.4 The Chain Rule (Thomas 14.4)

Chain Rule:

If $w = f(x, y)$ then

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

If $w = f(x, y, z)$ then

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

If $w = f(x, y)$, $x = g(r, s)$, $y = h(r, s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

If $w = f(x)$, $x = g(r, s)$ then

$$\frac{\partial w}{\partial r} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\partial x}{\partial r} \qquad \text{and} \qquad \frac{\partial w}{\partial s} = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\partial x}{\partial s}$$

Tree diagrams

Formula for implicit differentiation of $F(x, y)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

2.5 Directional Derivatives and Gradient Vectors (Thomas 14.5)

Definition of directional derivative, $(D_{\mathrm{u}}f)_{P_0}$.

Definition of gradient vector,

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Relationship with directional derivative:

$$D_{\mathrm{u}}f = \nabla f \cdot \mathrm{u}$$

The gradient of f is normal to the level curve.

2.6 Tangent Planes and Differentials (Thomas 14.6)

The tangent plane at $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ is the plane through P_0 normal to $\nabla f|_{P_0}$. Equation is

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

The normal line of the surface at P_0 is line through P_0 parallel to $\nabla f|_{P_0}$. Equation is

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t\,.$$

Linearisation

$$L(x, y, z) = f(x_0, y_0, z_0)+$$

$$+f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

Total differential resulting from change $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is:

$$\mathrm{d}f = f_x(x_0, y_0, z_0)\mathrm{d}x + f_y(x_0, y_0, z_0)\mathrm{d}y + f_z(x_0, y_0, z_0)\mathrm{d}z$$

2.7 Extreme Values and Saddle Points (Thomas 14.7)

Definitions of local minimum, local maximum, critical point, saddle point.

Second derivatives test for local maximum, local minimum, saddle point; when is test inconclusive.

Use of second derivatives and discriminant, $f_{xx}f_{yy} - f_{xy}^2$.

2.8 Lagrange Multipliers (Thomas 14.8)

Method of Lagrange Multipliers: To find the local maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ we find the values of x, y and z that simultaneously satisfy

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0\,.$$

3. MULTIPLE INTEGRALS

3.1 Double Integrals (Thomas 15.1, 15.2 + lecture notes)

Definition of double integral of function $f(x, y)$ over region R:

$$\int\!\!\int_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Connection with calculation of volume beneath surface using $z = f(x, y)$.

Iterated or repeated integral. Importance of order of integration; Fubini's theorem (two forms).

Method for double integrals: (i) sketch, (ii) find y-limits of integration, (iii) find x-limits of integration.

Bounded rectangular regions; bounded non-rectangular regions; unbounded regions.

Reversing order of integration; importance of finding new limits.

3.2 Area (Thomas 15.3 + lecture notes)

Area enclosed by a region R is the double integral,

$$\int\!\!\int_R \mathrm{d}x\,\mathrm{d}y\,.$$

Average value of a function $f(x, y)$ over a region R is

$$\frac{1}{\text{area of } R}\int\!\!\int_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

3.3 Change of Variables in Double Integrals

(Thomas 15.8, 15.4 + lecture notes)

Change of variables from (x, y) to, say (u, v).

Definition of Jacobian matrix:

$$\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

Definition of Jacobian (or Jacobian determinant):

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = (\partial x/\partial u)(\partial y/\partial v) - (\partial y/\partial u)(\partial x/\partial v)\,.$$

Using Jacobian of transformation from Cartesian to polar coordinates to get

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

Use of

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}\,.$$

Evaluation of

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x$$

by making it a double integral and then transforming to polar coordinates; connection with normal distribution and error function.

3.4 Triple Integrals (Thomas 15.5 + lecture notes)

Definition of triple integral of function $f(x, y, z)$ over volume V:

$$\int\int_V\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Volume enclosed by a volume V is the triple integral,

$$\int\int_V\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,.$$

Average value of a function $f(x, y, z)$ over a volume V is

$$\frac{1}{\text{volume of } V} \int\int_V\int f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,.$$

3.5 Change of Variables in Triple Integrals (Thomas 15.8 + lecture notes)

Change of variables from (x, y, z) to, say (u, v, w).

Definition of Jacobian matrix:

$$\begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{pmatrix}.$$

Definition of Jacobian (or Jacobian determinant):

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}.$$

4. INFINITE SEQUENCES AND SERIES

4.1 Sequences (Thomas 10.1)

Lists of numbers $\{a_n\}$

Convergence and divergence of sequences

Limit of a sequence

Sandwich Theorem for sequences

Continuous Function Theorem for sequences

Non-decreasing sequences

Sequences bounded from above

Upper bound

Least upper bound

4.2 Infinite Series (Thomas 10.2)

Sequence of partial sums

Convergent series and their sum; divergent series

Geometric series:

$$\sum_{n=0}^{\infty} a\,r^n = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} a\,r^{n-1}$$

(ratio r)

$$\sum_{n=1}^{\infty} a\,r^{n-1} = \frac{a}{1-r}$$

n-th Term Test for Divergence:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

4.3 The Integral Test (Thomas 10.3)

Divergence of the harmonic series, $\sum_{n=1}^{\infty} 1/n$

Convergence of the series, $\sum_{n=1}^{\infty} 1/n^2$

Integral test: If $a_n = f(n)$ then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,\mathrm{d}x$ both converge or both diverge (proof using graphs for the case $n = 1$).

4.4 Ratio Tests (Thomas 10.5)

Ratio Test:

If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

then (i) the series converges if $\rho < 1$, (ii) the series diverges if $\rho > 1$ or ρ is infinite and (iii) the test is inconclusive if $\rho = 1$.

4.5 Power Series (Thomas 10.7)

Power series about $x = 0$:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

Power series about $x = a$:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

Radius of convergence, interval of convergence.

Alternating Series Test:

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following hold:

The u_n's are all positive

$u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N

$u_n \to 0$

Absolute convergence

Conditional convergence

4.6 Taylor and Maclaurin Series (Thomas 10.8)

Taylor series generated by function f at $x = a$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Maclaurin series generated by function f at $x = 0$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k =$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Taylor polynomial of order n

4.7 Convergence of Taylor Series;

Error Estimates (Thomas 10.9)

Taylor's formula

Remainder R of order n (error term)

Remainder Estimation Theorem:

$$|R_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}$$

4.8 Applications of Power Series (Thomas 10.10)

Binomial series:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

Solving differential equations:

Assume a power series solution of the form,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

substitute in the differential equation and solve for the coefficients.

Evaluating non-elementary integrals

Evaluating indeterminate forms