Syllabus

The lecture course on *Superconductivity* will be given in 6 lectures in Trinity term.

1. Introduction to superconductivity.
2. The London equations
3. Ginzburg-Landau theory
4. The Josephson effect
5. BCS theory
6. Unconventional superconductors

Reading

• '*Superconductivity, Superfluids and Condensates*', by J. F. Annett, OUP 2004: the best book for the course.

• '*Solid State Physics*' N. W. Ashcroft and N. D. Mermin, chapters 34, is a good overview of some of the material in the course, though out of date on experiments and written in cgs units. The relevant chapters in the solid state texts by Kittel can also be consulted.

• An advanced but informative description of the ideas concerning broken symmetry may be found in the second chapter of '*Basic Notions of Condensed Matter Physics*', P. W. Anderson (Benjamin-Cummings 1984). For enthusiasts only!

• A popular account of the history of superconductivity can be found in my own '*Superconductivity: A Very Short Introduction*', OUP 2009.

Web-page for the course: `http://users.ox.ac.uk/~sjb/supercond.html`

Maxwell's equations:

In free space, Maxwell's equations are

$$\nabla \cdot \boldsymbol{\mathcal{E}} = \rho/\epsilon_0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t}, \tag{4}$$

and describe the relationships between the electric field $\boldsymbol{\mathcal{E}}$, the magnetic induction $\mathbf{B}$, the charge density ρ and the current density $\mathbf{J}$. Equation 1 shows that electric field diverges away from positive charges and converges into negative charges; charge density therefore acts as a source or a sink of electric field. Equation 2 shows that magnetic fields have no such divergence; thus there are no magnetic charges (monopoles) and lines of $\mathbf{B}$ field must just exist in loops; they can never start or stop anywhere. Equation 3 shows that you only get loops of electric field around regions in space in which there is a changing magnetic field. This leads to Faraday's law of electromagnetic induction. Equation 4, in the absence of a changing electric field, shows that loops of magnetic induction are found around electric currents. In the presence of matter, we have:

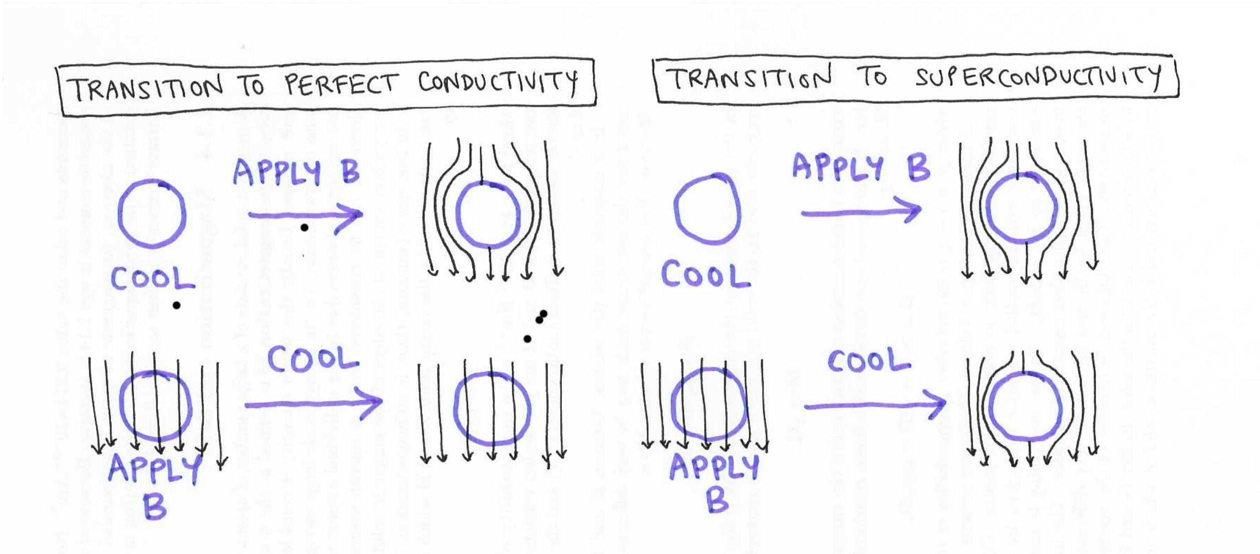
$$\nabla \cdot \mathbf{D} = \rho_{\text{free}} \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7}$$

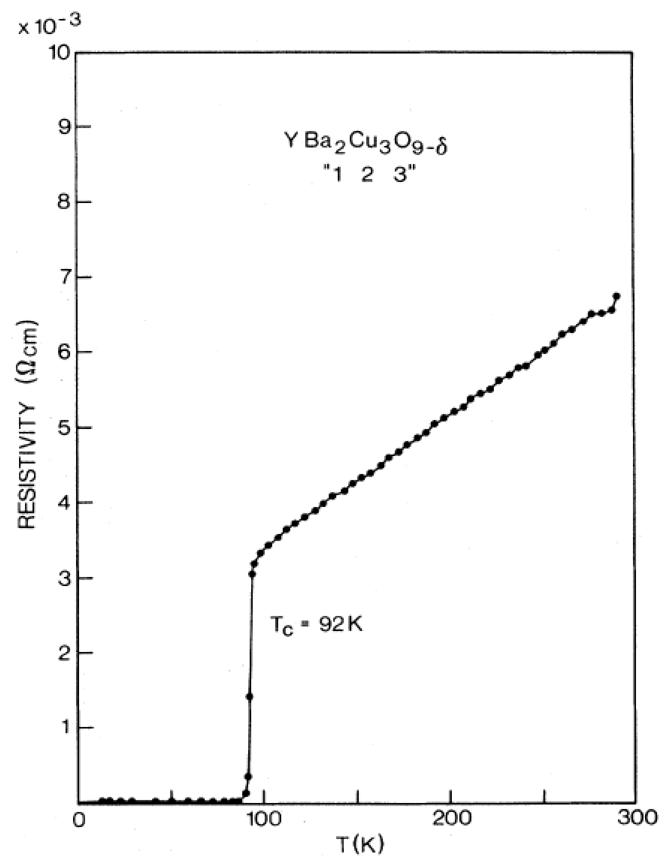
$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t}, \tag{8}$$

Meissner effect



Superconducting transition temperatures

Substance	T_c (K)
Al	1.196
Hg	4.15
In	3.40
Nb	9.26
Pb	7.19
Sn	3.72
Zn	0.875
Nb_3Sn	18.1
V_3Si	17
Nb_3Ge	23.2
$BaPbO_3$	0.4
$Ba_xLa_{5-x}Cu_5O_y$	30–35
$YBa_2Cu_3O_{7-\delta}$	95
$Bi_2Sr_2Ca_2Cu_3O_{10}$	110
$Hg_{0.8}Pb_{0.2}Ba_2Ca_2Cu_3O_x$	133
$HgBa_2Ca_2Cu_3O_{8+\delta}$ at 30 GPa	164
URu_2Si_2	1.3
MgB_2	39
YNi_2B_2C	12.5
$(TMTSF)_2ClO_4$	1.4
K_3C_{60}	19
Cs_3C_{60} at 7 kbar	38
$(BEDT\text{-}TTF)_2Cu(NCS)_2$	10.4
$(BEDT\text{-}TTF)_2Cu[N(CN)_2]Br$	11.8
$Sm[O_{1-x}F_x]FeAs$	55



Thermodynamics

At constant pressure $\mathrm{d}G = -S\,\mathrm{d}T - m\,\mathrm{d}B$. Therefore, when T is constant and less than T_C

$$G_\mathrm{s}(B) - G_\mathrm{s}(0) = -\int_0^B m\,\mathrm{d}B. \tag{9}$$

Here $m = MV$ and $M = -H = -B/\mu_0$. This implies that

$$G_\mathrm{s}(B) - G_\mathrm{s}(0) = -\int_0^B m\,\mathrm{d}B = \frac{VB^2}{2\mu_0}. \tag{10}$$

At $B = B_\mathrm{c}$, we have that

$$G_\mathrm{s}(B_\mathrm{c}) = G_\mathrm{n}(B_\mathrm{c}) = G_\mathrm{n}(0) \tag{11}$$

because

- The superconducting and normal states are in equilibrium
- We assume no field dependence in G_n

Hence

$$G_\mathrm{n}(0) - G_\mathrm{s}(0) = \frac{VB^2}{2\mu_0}, \tag{12}$$

and therefore

$$S_\mathrm{n} - S_\mathrm{s} = -\frac{V}{\mu_0} B_\mathrm{c} \frac{\mathrm{d}B_\mathrm{c}}{\mathrm{d}T} > 0, \tag{13}$$

because

$$\frac{\mathrm{d}B_\mathrm{c}}{\mathrm{d}T} < 0. \tag{14}$$

Therefore the *entropy of the superconducting state is lower than the normal state.* Differentiation yields

$$C_\mathrm{n} - C_\mathrm{s} = -\frac{TV}{\mu_0}\left[B_\mathrm{c}\frac{\mathrm{d}^2 B_\mathrm{c}}{\mathrm{d}T^2} + \left(\frac{\mathrm{d}B_\mathrm{c}}{\mathrm{d}T}\right)^2\right]. \tag{15}$$

London equation

Fritz London, working with Heinz London, realised that superconductivity was due to a **macroscopic quantum phenomenon** in which there was long range order of the momentum vector. This implies **condensation in momentum space**. Fritz London also realised that it is the rigidity of the superconducting wave function ψ which is responsible for diamagnetism.

The London equation is

$$\boxed{\mathbf{J} = -\frac{nq^2}{m}\mathbf{A}} \tag{16}$$

This leads to an equation for the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ of the form

$$\boxed{\nabla^2 \mathbf{B} = \frac{\mathbf{B}}{\lambda^2}} \tag{17}$$

This differential equation can be solved in various geometries.

A reminder: Canonical momentum

In classical mechanics the Lorentz force $\mathbf{F}$ on a particle with charge q moving with velocity $\mathbf{v}$ in an electric field $\boldsymbol{\mathcal{E}}$ and magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q(\boldsymbol{\mathcal{E}} + \mathbf{v} \times \mathbf{B}). \tag{18}$$

Using $\mathbf{F} = m\mathrm{d}\mathbf{v}/\mathrm{d}t$, $\mathbf{B} = \nabla \times \mathbf{A}$ and $\boldsymbol{\mathcal{E}} = -\nabla V - \partial \mathbf{A}/\partial t$, where V is the electric potential, $\mathbf{A}$ is the magnetic vector potential and m is the mass of the particle, eqn 18 may be rewritten as

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -q\nabla V - q\frac{\partial \mathbf{A}}{\partial t} + q\mathbf{v} \times (\nabla \times \mathbf{A}). \tag{19}$$

The vector identity

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{20}$$

can be used to simplify eqn 19 leading to

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + q\left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}\right) = -q\nabla(V - \mathbf{v} \cdot \mathbf{A}). \tag{21}$$

Note that $m\,\mathrm{d}\mathbf{v}/\mathrm{d}t$ is the force on a charged particle measured in a coordinate system that moves with the particle. The partial derivative $\partial \mathbf{A}/\partial t$ measures the rate of change of $\mathbf{A}$ at a fixed point in space. We can rewrite eqn 21 as

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v} + q\mathbf{A}) = -q\nabla(V - \mathbf{v} \cdot \mathbf{A}) \tag{22}$$

where $\mathrm{d}\mathbf{A}/\mathrm{d}t$ is the *convective derivative* of $\mathbf{A}$, written as

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}, \tag{23}$$

which measures the rate of change of $\mathbf{A}$ at the location of the moving particle. Equation 22 takes the form of Newton's second law (i.e. it reads 'the rate of change of a quantity that looks like momentum is equal to the gradient of a quantity that looks like potential energy'). We therefore define the *canonical momentum*

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} \tag{24}$$

and an effective potential energy experienced by the charge particle, $q(V - \mathbf{v} \cdot \mathbf{A})$, which is velocity-dependent. The canonical momentum reverts to the familiar momentum $m\mathbf{v}$ in the case of no magnetic field, $\mathbf{A} = 0$. The kinetic energy remains equal to $\frac{1}{2}mv^2$ and this can therefore be written in terms of the canonical momentum as $(\mathbf{p} - q\mathbf{A})^2/2m$.

Gauge symmetry

The relationship between fields $\mathbf{E}$ and $\mathbf{B}$ and potentials ϕ and $\mathbf{A}$ is

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{25}$$
$$\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}. \tag{26}$$

A scalar function χ can allow one to alter the potentials using

$$\mathbf{A} \to \mathbf{A} + \nabla\chi \tag{27}$$
$$\phi \to \phi - \dot{\chi} \tag{28}$$

and $\mathbf{E}$ and $\mathbf{B}$ are unaltered. A choice of χ is called a choice of gauge. However, the London equation $\mathbf{J} = -\frac{nq^2}{m}\mathbf{A}$ only works in one choice of gauge, known as the London gauge. The continuity equation $\dot{\rho} + \nabla \cdot \mathbf{J} = 0$ in the DC case is just $\nabla \cdot \mathbf{J} = 0$ and so the London gauge amounts to choosing $\nabla \cdot \mathbf{A} = 0$. Notice that the momentum $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$ is therefore not gauge invariant either:

$$\mathbf{p} \to \mathbf{p} + q\nabla\chi \tag{29}$$

If a wave function has a phase θ which depends on position in space, i.e. $\psi(\mathbf{r}) = \psi \mathrm{e}^{i\theta(\mathbf{r})}$, then since $\mathbf{p} = -i\hbar\nabla$ and

$$-i\hbar\nabla \mathrm{e}^{i\theta(\mathbf{r})} = \hbar\nabla\theta \mathrm{e}^{i\theta(\mathbf{r})}, \tag{30}$$

then we see that this phase (and hence the wave function) is also not gauge invariant. If

$$\theta \to \theta + \frac{q\chi}{\hbar}, \tag{31}$$

then $m\mathbf{v}$ is gauge invariant. Note here that

$$m\mathbf{v} = \hbar\nabla\theta - q\mathbf{A}, \tag{32}$$

and so the effect of the gauge transformations on θ and $\mathbf{A}$ cancel out.

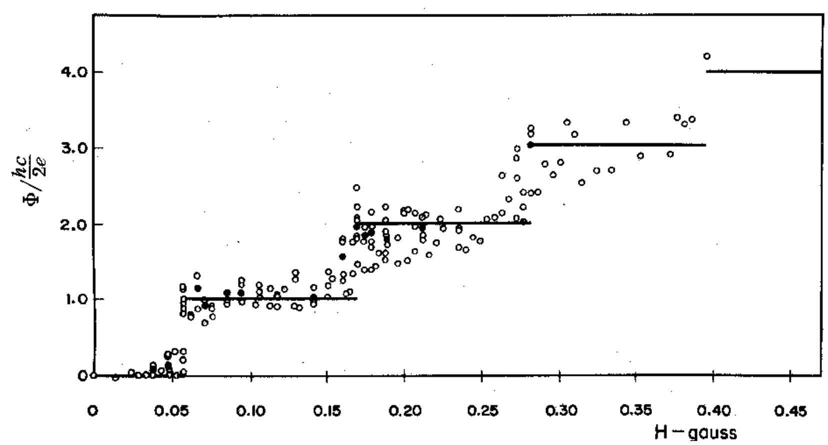
Flux quantization

Flux quantization leads to the equation $\Phi = N\Phi_0$ where N is an integer and Φ_0 is the **flux quantum**:

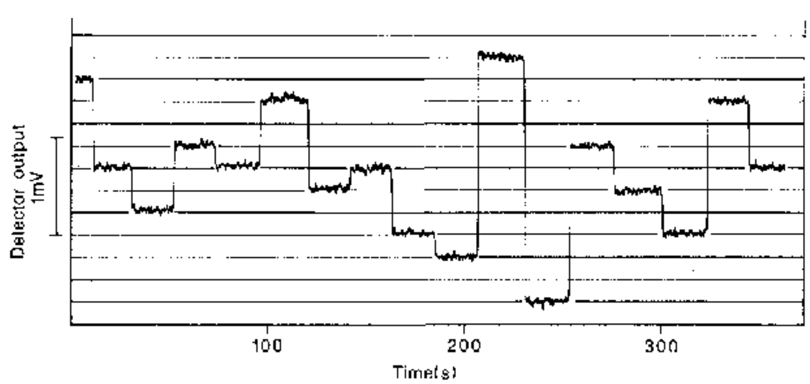
$$\Phi_0 = \frac{h}{2e}. \tag{33}$$

The $2e$ in this equation represents the charge of the superconducting carrier, and experiment implies that the carrier consists of a **pair** of electrons.

The first evidence for this pairing came from the data shown below [B. S. Deaver and W. M. Fairbank, Phys. Rev. Lett. **7**, 43 (1961)]; this is from an experiment on a cylinder made of tin (Sn). Note the quaint oldy-woldy units.



The experiment shown below [C. E. Gough et al, Nature **326**, 855 (1987)] tried a similar experiment, this time using a ring made out of a high-T_c superconductor ($Y_{1.2}Ba_{0.8}CuO_4$). What is shown here is the output of an rf-SQUID magnetometer. The ring was exposed to a source of electromagnetic noise so that the flux varied through the ring. Once the output of the rf-SQUID was calibrated, one could show that the flux jumps were 0.97 ± 0.04 ($h/2e$), confirming that the charge carriers in the high-T_c materials were pairs of electrons.



Ginzburg-Landau theory

In the lectures, we will motivate the Ginzburg-Landau expression:

$$F_{\mathrm{s}} = F_{\mathrm{n}} + \int \mathrm{d}^3 r \left[a(T)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{1}{2m}|-i\hbar\nabla\psi + 2e\mathbf{A}\psi|^2 + \frac{(B-B_0)^2}{2\mu_0} \right] \tag{34}$$

This yields expressions for the penetration depth λ:

$$\lambda = \sqrt{\frac{mb}{4\mu_0 e^2 |a(T)|}} \tag{35}$$

and the coherence length ξ:

$$\xi = \sqrt{\frac{\hbar^2}{2m|a(T)|}}. \tag{36}$$

Part of the derivation from the lectures is included here: When the magnetic field can be ignored (and setting $A = 0$), we have

$$F_{\mathrm{s}} = F_{\mathrm{n}} + \int \mathrm{d}^3 r\, f, \tag{37}$$

where $f = a(T)|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{\hbar^2}{2m}|\nabla\psi|^2$. If ψ is varied, then

$$\mathrm{d}f = 2a\psi\,\mathrm{d}\psi + 2b\psi^3\,\mathrm{d}\psi + \frac{\hbar^2}{2m}\mathrm{d}|\nabla\psi|^2, \tag{38}$$

and $\mathrm{d}|\nabla\psi|^2 = |\nabla(\psi+\mathrm{d}\psi)|^2 - |\nabla\psi|^2 = 2\nabla\psi\cdot\nabla(\mathrm{d}\psi)$. In the integral, the term $\nabla\cdot[\nabla\psi\,\mathrm{d}\psi] = (\nabla^2\psi)\,\mathrm{d}\psi + \nabla\psi\cdot\mathrm{d}\nabla\psi$ gives a surface contribution, and hence

$$\mathrm{d}f = 2\mathrm{d}\psi[(a+b\psi^2)\psi - \frac{\hbar^2}{2m}\nabla^2\psi] = 0 \tag{39}$$

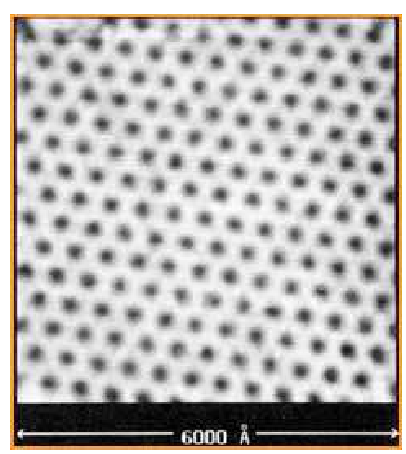
for any ψ. This looks like a non-linear Schrödinger equation. Near T_{c} we can neglect the $b\psi^2$ term because $\psi \to 0$ and then the equation takes the form

$$\nabla^2\psi = \frac{\psi}{\xi^2} \tag{40}$$

where $\xi = \sqrt{\frac{\hbar^2}{2m|a(T)|}}$.

The Ginzburg-Landau parameter κ is defined by $\kappa = \lambda/\xi$. If $\kappa < 1/\sqrt{2}$, we have a type I superconductor. If $\kappa > 1/\sqrt{2}$, we have a type II superconductor.

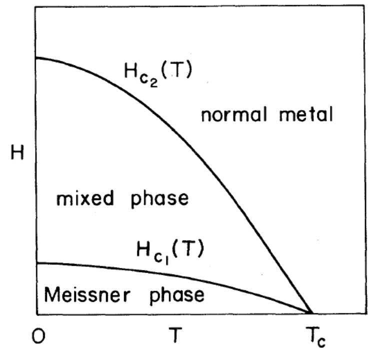
The figure shows the vortex lattice imaged in $NbSe_2$ (a type II superconductor with a transition temperature of 7.2 K and a critical field of 3.2 T) by tunneling into the superconducting gap edge with a low-temperature scanning-tunneling microscope. The magnetic field used is 1 T. H. F. Hess, R. B. Robinson, R. C. Dynes, J. M. Valles, Jr., and J. V. Waszczak, Phys. Rev. Lett. **62**, 214 (1989).



For a square vortex lattice of spacing d, we have that $\Phi_0 = Bd^2$ and so $d = (\Phi_0/B)^{1/2}$. For a triangular vortex lattice $d = (2\Phi_0/\sqrt{3}B)^{1/2}$.

Type II superconductors

The Ginzburg-Landau parameter κ is defined by $\kappa = \lambda/\xi$ and if $\kappa > 1/\sqrt{2}$, we have a type II superconductor. In this case vortices will form into a lattice for fields between B_{c1} and B_{c2}. The phase diagram is shown below.



Silsbee's rule

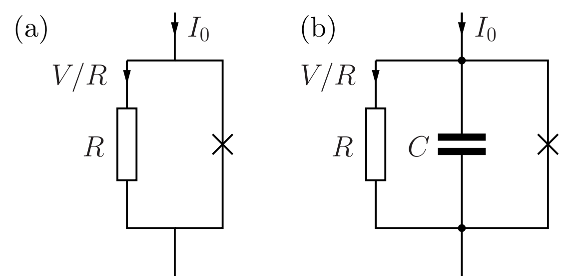
For a wire of radius a, the critical current is related to the critical field by $I_c = 2\pi a B_c/\mu_0$.

The Josephson effect

- The DC Josephson effect: $I = I_J \sin\phi$, where ϕ is the phase difference across the Josephson junction.
- The AC Josephson effect: $\hbar\dot{\phi} = 2eV$ so that $I = I_J \sin(\omega_J t + \phi_0)$ where $\omega_J = 2eV/\hbar$.
- The inverse AC Josephson effect: For an ac voltage $V = V_0 + V_{rf}\cos\omega t$, we have that

$$
\begin{aligned}
I &= I_J \sin(\omega_J t + \phi_0 + \frac{2eV_{rf}}{\hbar\omega}\sin\omega t) \\
I &= I_J \sum_{n=-\infty}^{\infty} (-1)^n J_n\left(\frac{2eV_{rf}}{\hbar\omega}\right)\sin[(\omega_J - n\omega)t + \phi_0]
\end{aligned}
$$

A perfect Josephson junction (signified by the cross) can be inserted in various electrical circuits. (A real Josephson junction may well have some real resistance or capacitance so this circuit can be thought of as an attempt to model real junctions.)



- The resistively shunted Josephson (RSJ) model yields

$$I_0 = I_J \sin\phi + \frac{V}{R} = I_J \sin\phi + \frac{\hbar\dot{\phi}}{2eR}. \tag{41}$$

Adding in a capacitor gives

$$I_0 = I_J \sin\phi + \frac{V}{R} + C\dot{V} = I_J \sin\phi + \frac{\hbar\dot{\phi}}{2eR} + \frac{\hbar C\ddot{\phi}}{2e}. \tag{42}$$

This can be rewritten as

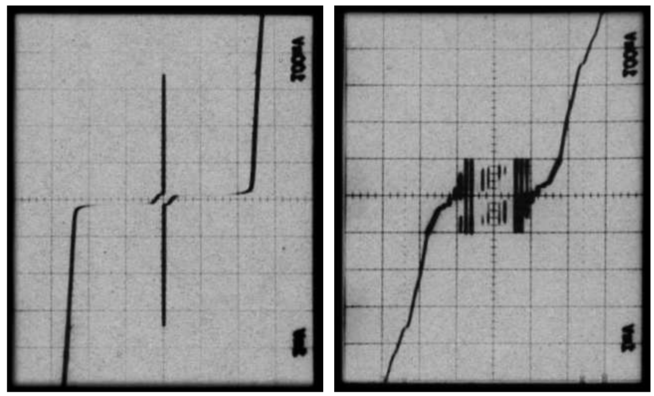
$$m\ddot{\phi} = -\frac{\partial U}{\partial\phi} - \frac{\hbar}{2eR}\dot{\phi}, \tag{43}$$

where $m = \hbar C/2e$ and $U = -I_J\cos\phi - I_0\phi$ is the *tilted washboard potential.*

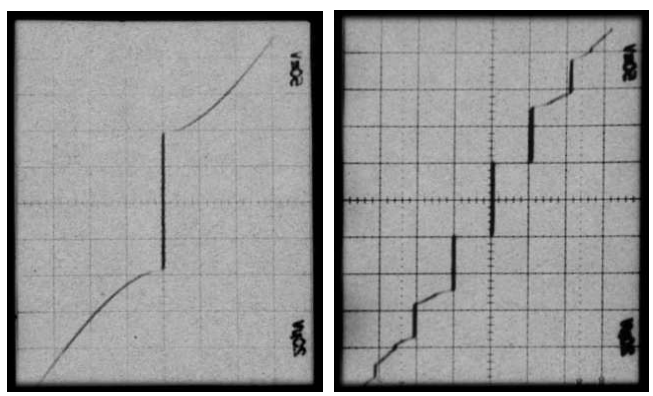
- The gauge invariant phase difference is written as

$$\phi = \theta_1 - \theta_2 - \frac{2e}{\hbar}\int_1^2 \mathbf{A}\cdot\mathbf{dl} \tag{44}$$

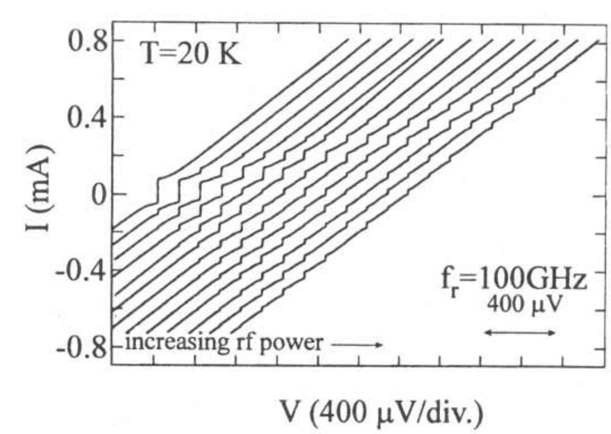
This expression is used to explain the behaviour of the SQUID Superconducting Quantum Interference Device.



Strongly underdamped superconductor-insulator-superconductor (SIS) junction. The current-voltage characteristic (I horizontal, V vertical) for a Nb–Al_2O_3 –Nb junction [left] without and [right] with microwave radiation of 70 GHz.



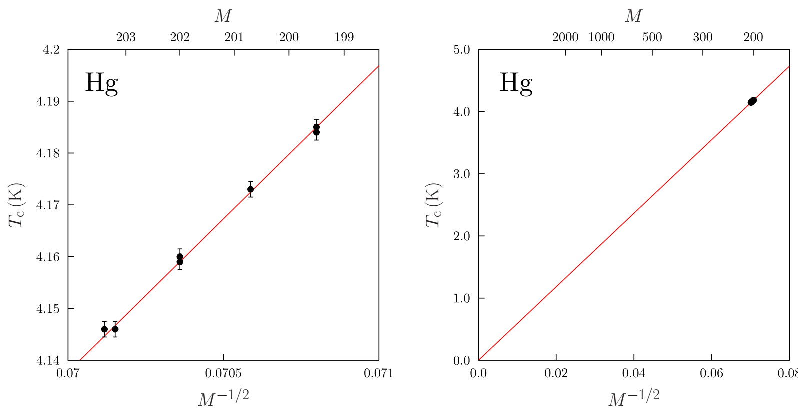
Highly damped superconductor-normal-superconductor (SNS) junction. The current-voltage characteristic (I horizontal, V vertical) for a Nb–PdAu–Nb junction [left] without and [right] with microwave radiation of 10 GHz.



The current-voltage characteristic (I horizontal, V vertical) for a high-T_c Josephson junction under microwave illumination. Data are shown for increasing microwave power.

The isotope effect

The transition temperature $T_\mathrm{c} \propto M^{-1/2}$ where M is the mass of the isotope.



This is very good evidence for the role of *phonons* in superconductivity.

Creation and annihilation operators

We define a creation operator $\hat{a}^\dagger$ and an annihilation operator $\hat{a}$ for the harmonic oscillator problem:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2. \tag{45}$$

Since $[\hat{x}, \hat{p}] = i\hbar$ we have that $[\hat{a}, \hat{a}^\dagger] = 1$. Furthermore, we can write

$$\begin{aligned} \hat{a}^\dagger |n\rangle &= \sqrt{n+1}|n+1\rangle & (46) \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle & (47) \\ \hat{a}^\dagger \hat{a}|n\rangle &= n|n\rangle, & (48) \end{aligned}$$

and hence $\hat{a}^\dagger \hat{a}$ is the number operator. The Hamiltonian becomes

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}), \tag{49}$$

and the eigenvalues are $E = (n + \frac{1}{2})\hbar\omega$. Note that

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n |0\rangle. \tag{50}$$

Coherent states

A coherent state $|\alpha\rangle$ is defined by

$$|\alpha\rangle = C\left[|0\rangle + \frac{\alpha}{\sqrt{1!}}|1\rangle + \frac{\alpha^2}{\sqrt{2!}}|2\rangle + \frac{\alpha^3}{\sqrt{3!}}|3\rangle + \cdots\right], \tag{51}$$

where $\alpha = |\alpha|\mathrm{e}^{i\theta}$ is a complex number. Hence

$$|\alpha\rangle = C\left[1 + \frac{\alpha\hat{a}^\dagger}{\sqrt{1!}} + \frac{(\alpha\hat{a}^\dagger)^2}{\sqrt{2!}} + \frac{(\alpha\hat{a}^\dagger)^3}{\sqrt{3!}} + \cdots\right]|0\rangle, \tag{52}$$

This state can be written $\boxed{|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\exp(\alpha\hat{a}^\dagger)|0\rangle}$. The coherent state is an eigenstate of the annihilation operator, so that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and has a well-defined phase but an uncertain number of particles.

- $\hat{c}^\dagger_{\mathbf{k}\sigma}$ is a creation operator for an electron with momentum $\mathbf{k}$ and spin σ.
- $\hat{c}_{\mathbf{k}\sigma}$ is a annihilation operator for an electron with momentum $\mathbf{k}$ and spin σ.
- $\hat{P}^\dagger_k = \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow}$ is a pair creation operator.

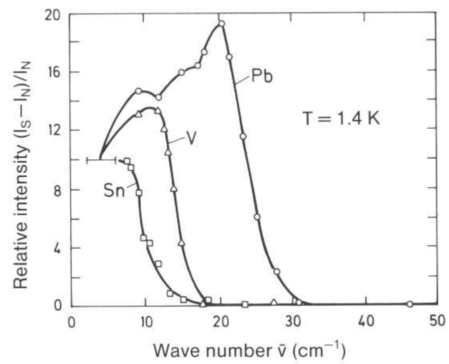
Note that we can write the Fermi sea as

$$|\text{Fermi sea}\rangle = \prod_{k<k_\text{F}} \hat{P}^\dagger_{\mathbf{k}\sigma}|0\rangle. \tag{53}$$

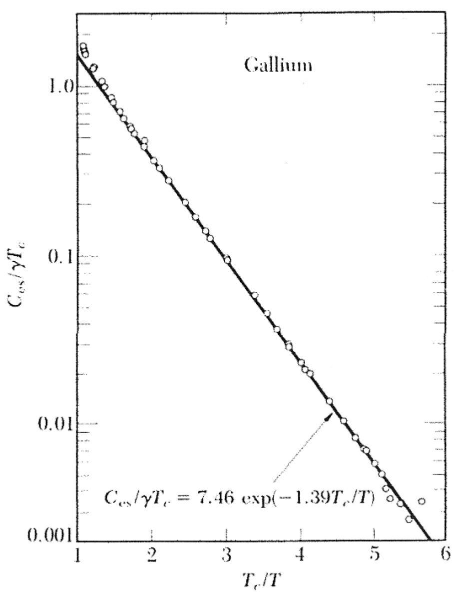
The BCS wave function will be written as a product of coherent states of pairs:

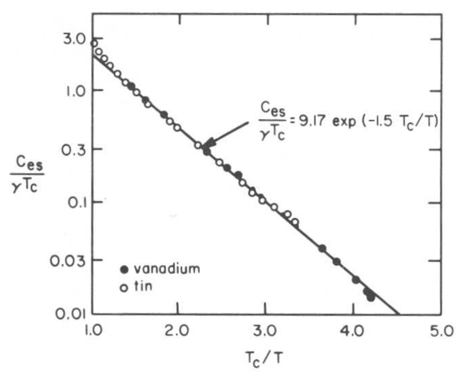
$$|\Psi_\text{BCS}\rangle = \text{constant} \times \prod_k \exp(\alpha_k \hat{P}^\dagger_k)|0\rangle. \tag{54}$$

Absorption of infra-red radiation

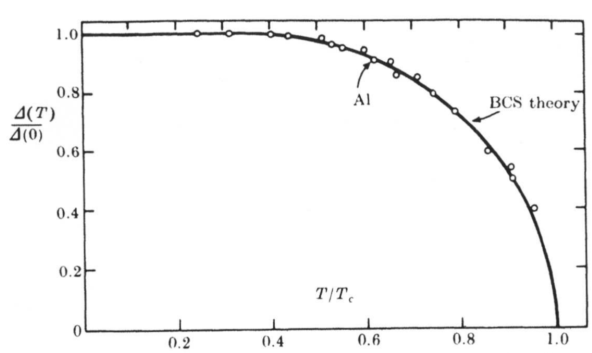


Heat capacity data





Temperature dependence of the gap



BCS theory

The derivation of BCS theory is rather involved and only an outline is given here. For more details, consult the books by Annett and by Schrieffer. This material is provided in this handout *for interest only* and the less-interested reader need only focus on the boxed results.

The BCS trial wave function can be written as a product of coherent states of pairs (we will drop the hats from the operators now):

$$|\Psi_{\rm BCS}\rangle = \text{constant} \times \prod_{\mathbf{k}} \exp(\alpha_{\mathbf{k}} P^\dagger_{\mathbf{k}})|0\rangle, \tag{55}$$

where $P^\dagger_{\mathbf{k}}$ is a pair creation operator. In the lecture, we showed that this could be written

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} P^\dagger_{\mathbf{k}})|0\rangle, \tag{56}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are variational parameters which can be adjusted to minimise the energy.

The BCS Hamiltonian is

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - |g_{\rm eff}|^2 \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}, \tag{57}$$

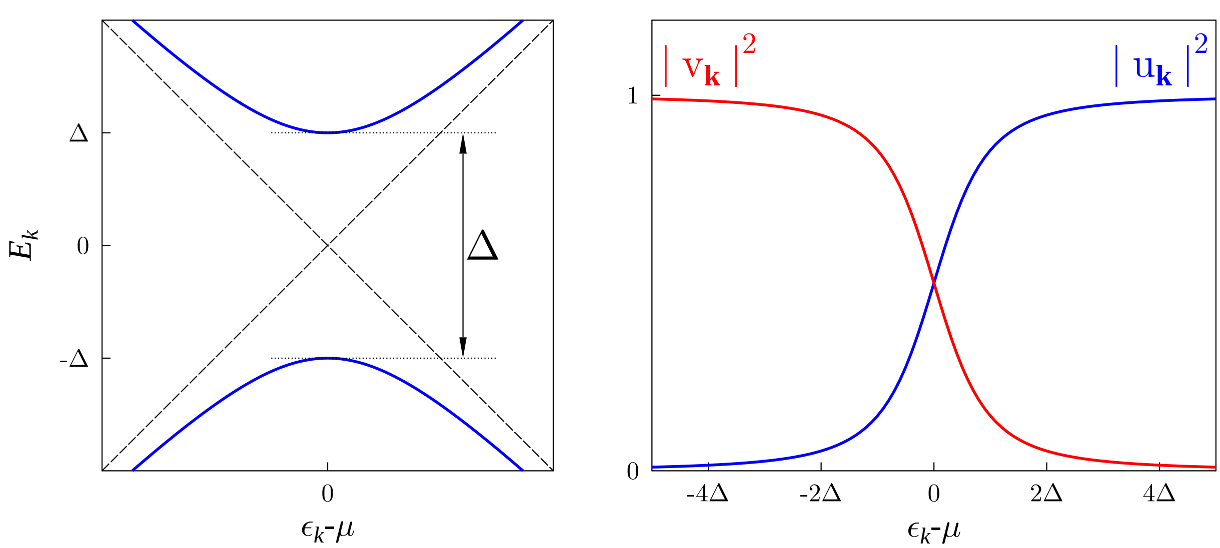
where the first term represents the kinetic energy and the second term accounts for the electron phonon interaction. This Hamiltonian is applied to the BCS Hamiltonian, and the energy is minimised with respect to the parameters $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$. The results of this are that

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right) \tag{58}$$

$$|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right) \tag{59}$$

$$E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + |\Delta|^2}. \tag{60}$$

These results are plotted in the graphs below. $E_{\mathbf{k}}$ can be interpreted as the energy of an electronic excitation (note that both electron and hole solutions emerge). The minimum electronic excitation energy is the energy gap Δ.



The gap parameter is defined by $\Delta = |g_{\text{eff}}|^2 \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}^*$. One also finds that $u_{\mathbf{k}} v_{\mathbf{k}}^* = \Delta/(2E_{\mathbf{k}})$. These two equations lead to the **BCS gap equation** at $T = 0$:

$$\Delta = |g_{\text{eff}}|^2 \sum_{\mathbf{k}} \frac{\Delta}{2E_{\mathbf{k}}}. \tag{61}$$

Writing $\lambda = |g_{\text{eff}}|^2 g(E_{\text{F}})$, where $g(E_{\text{F}})$ is the density of states at the Fermi level, this becomes

$$\Delta = \lambda \int_0^{\hbar\omega_{\text{D}}} \frac{\Delta \, \mathrm{d}\epsilon}{\sqrt{\Delta^2 + \epsilon^2}}, \tag{62}$$

and so

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_{\text{D}}} \frac{\mathrm{d}\epsilon}{\sqrt{\Delta^2 + \epsilon^2}} = \sinh^{-1}\left(\frac{\hbar\omega_{\text{D}}}{\Delta}\right). \tag{63}$$

Since $\Delta \ll \hbar\omega_{\text{D}}$, we have that $e^{1/\lambda}/2 \approx \hbar\omega_{\text{D}}/\Delta$ and so

$$\boxed{\Delta \approx 2\hbar\omega_{\text{D}} e^{-1/\lambda}}. \tag{64}$$

When $T \neq 0$, we must replace eqn 62 by

$$\Delta = \lambda \int_0^{\hbar\omega_{\text{D}}} \frac{\Delta \, \mathrm{d}\epsilon}{\sqrt{\Delta^2 + \epsilon^2}} [1 - 2f(\epsilon)], \tag{65}$$

where $f(\epsilon)$ is the Fermi-Dirac function. Hence, for $T = T_{\text{c}}$ we have that $\Delta = 0$ and so

$$\frac{1}{\lambda} = \int_0^{x_{\text{D}}} \frac{\tanh x}{x} \mathrm{d}x, \tag{66}$$

where $x_{\text{D}} = \hbar\omega_{\text{D}}/2k_{\text{B}}T_{\text{c}}$ and hence

$$\boxed{k_{\text{B}}T_{\text{c}} = 1.13\hbar\omega_{\text{D}} e^{-1/\lambda}}. \tag{67}$$

Eqns 64 and 67 can be combined to yield

$$\boxed{2\Delta(0) = 3.52 k_{\text{B}}T_{\text{c}}}. \tag{68}$$

Experimental values are given in the following table:

Material	$2\Delta(0)/k_{\text{B}}T_{\text{c}}$
Zn	3.2
Al	3.4
In	3.6
Hg	4.6
Pb	4.3
Nb	3.8
K_3C_{60}	3.6
$YBa_2Cu_3O_{7-\delta}$	4.0