Vector analysis and vector identities by means of cartesian tensors

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1 The cartesian tensor concept

1.1 Introduction

The cartesian tensor approach to vector analysis uses components in a rectangular coordinate system to derive all vector and field relationships. These relationships may then be transformed to other coordinate systems and expressed in coordinate-free vector notation. The result is much simpler than attempting derivations in a coordinate-free manner or in multiple coordinate systems.

Vector identities and vector differential operations and integral theorems often appear complicated in standard vector notation. But by means of cartesian tensors the derivations of these can be unified, simplified, and made accessible to all students.

Typical undergraduate electromagnetic theory texts do not mention the cartesian tensor method, and modern advanced texts usually do not include vector analysis as a topic but only summarize its results. I learned the cartesian tensor approach from Professor Charles Halijak in a class at Kansas State University in 1962. I have since found the method, or parts of it, expounded in texts on vectors and tensors. The earliest reference I have read is by Levi-Civita[1]. His work, in a slightly modified form, is included in a text on tensor analysis by Sokolnikoff[2] and is used in a text on the theory of relativity by Møller[3]. The ideas in the following are also given in a "programmed instruction" text[4] and all of the following relationships are derived in one section or another of the text by Bourne and Kendall[5]. I will not give any further references in the following, but will present the work in a tutorial manner.

1.2 Definitions and notation

Typically, rectangular cartesian coordinates use x, y, and z as the variables for distances along the axes. In the cartesian tensor method these are replaced by x_1, x_2, and x_3 with an unspecified coordinate given as x_i. If unit vectors are denoted by the coordinates with "hats" over them: $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, then in the usual vector notation, a vector field $\mathbf{A}$ is expressed as

$$\mathbf{A} = A_1\hat{x}_1 + A_2\hat{x}_2 + A_3\hat{x}_3. \tag{1}$$

With the use of subscripts to identify axes, this component form may be written as a sum:

$$\mathbf{A} = \sum_{1}^{3} A_i \hat{x}_i. \tag{2}$$

Rather than keep all the notation of eq.(2), in the cartesian tensor method one represents the vector field by A_i, just as one represents the coordinates by x_i.

As an example of the simplification this notation gives, consider the dot product of two vectors, **A** and **B**:

$$\mathbf{A} \cdot \mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = \sum_{1}^{3} A_i B_i \tag{3}$$

This sum is simplified even further by writing for the dot product the term $A_i B_i$ where summation on repeated subscripts is assumed. Thus we make the identification

$$\mathbf{A} \cdot \mathbf{B} \longleftrightarrow A_i B_i \tag{4}$$

This is known as the "Einstein summation convention." When two subscripts in a term in an expression are the same then the term is to be summed on the repeated subscript over the range of the subscript. (This extends to higher dimensioned vector spaces, *e.g.* the four-vector space of special relativity – hence Einstein's use of it.)

1.3 Vectors and tensors

- An object having one subscript is called a *vector*, for example A_i.
- An object having no subscript is called a scalar, for example c.
- An object having multiple subscripts is called a tensor, and the *rank* of the tensor is the number of subscripts: for example, m_{im} is a second rank tensor. Vectors are first rank tensors, and scalars are zero rank tensors.

The concept of being a tensor involves more than having subscripts. A tensor is an object which transforms on a rotation of coordinates like a direct product of as many vectors as its rank. *Direct product* means just multiplying the objects together. Thus if A_i and B_i are vectors, $A_i B_j$ is a second rank tensor. Setting these indices equal (and summing on them) removes the dependence on the subscript with the result being a scalar, or zero rank tensor. Setting two subscripts of a tensor to the same letter and summing over its values reduces the rank of the tensor by two. This is called *contraction*.

1.4 Special tensors

There are two special tensors which are needed in deriving vector identities. These are the Kronecker symbol, δ_{ij}, and the Levi-Civita symbol, ϵ_{ijk}.

The Kronecker symbol is usually referred to as the "Kronecker delta" and it is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{5}$$

The δ_{ij} are the matrix elements of the identity matrix. The definition of δ_{ij} is such that the contraction of it with another tensor causes a substitution of the subscript letter from the one in the Kronecker delta to the one in the other tensor:

$$\delta_{ij} b_{kjl} = b_{kil} \tag{6}$$

for example.

The Levi-Civita symbol is also called the *alternating tensor* and is the totally antisymmetric tensor of the same rank as the number of dimensions. For three dimensions it is

$$\epsilon_{ijk} = \begin{cases} 1, & \text{for } i, j, k \text{ an even permutation of } 1, 2, 3 \\ -1, & \text{for } i, j, k \text{ an odd permutation of } 1, 2, 3 \\ 0, & \text{for any two of the subscripts equal} \end{cases} \tag{7}$$

The definition of the Levi-Civita symbol is closely related to the definition of determinants. In fact, determinants could be defined in terms of the ϵ_{ijk}. If a square matrix A has elements a_{ij} then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{8}$$

where summation on the repeated subscripts is implied.

1.5 Relations between the special tensors

The definitions of ϵ_{ijk} and δ_{ij} given in eqs.(7) and (5) result in the following identity

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \tag{9}$$

where summation is implied over index k. This identity may be demonstrated from the properties of determinants. The demonstration is not difficult, but it is lengthy and so is placed in Appendix A.

2 Vector operations and vector identities

With the Levi-Civita symbol one may express the vector cross product in cartesian tensor notation as:

$$\mathbf{A} \times \mathbf{B} \longleftrightarrow \epsilon_{ijk} A_j B_k. \tag{10}$$

This form for cross product, along with the relationship of eq.(9), allows one to form vector identities for repeated dot and cross products. For example, the triple vector product becomes

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \longleftrightarrow \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m. \tag{11}$$

The tensor side of this equation is evaluated by means of eq.(9) as

$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_j B_l C_m = B_i A_j C_j - C_i A_j B_j$$

to establish the standard identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

The del operator is represented in cartesian tensors by

$$\nabla \longleftrightarrow \partial/\partial x_i \tag{12}$$

so that

$$\begin{aligned} \nabla \mathbf{f} \quad &\longleftrightarrow \quad \partial f/\partial x_i & (13)\\ \nabla \cdot \mathbf{A} \quad &\longleftrightarrow \quad \partial A_i/\partial x_i & (14)\\ \nabla \times \mathbf{A} \quad &\longleftrightarrow \quad \epsilon_{ijk}\partial A_k/\partial x_j & (15) \end{aligned}$$

become the gradient, divergence, and curl. Identities involving the del operator may be evaluated in a manner similar to that done for the triple vector product above. For example

$$(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) \longleftrightarrow \epsilon_{ijk}\epsilon_{ilm}(\partial A_k/\partial x_j)(\partial B_m/\partial x_l). \tag{16}$$

The tensor side of this equation may be simplified as

$$(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(\partial A_k/\partial x_j)(\partial B_m/\partial x_l) = (\partial A_k/\partial x_j)(\partial B_k/\partial x_j) - (\partial A_k/\partial x_j)(\partial B_j/\partial x_k).$$

This example yields an identity which cannot be expressed in terms of ordinary vector notation, but requires dyadic notation. As such it is not usually mentioned in beginning courses on vectors.

3 Integral theorems

The divergence theorem, also known as Gauss's theorem, is widely used. In conventional vector notation it is

$$\oint\!\!\oint \mathbf{D} \cdot d\mathbf{S} = \iiint \nabla \cdot \mathbf{D}\, dv. \tag{17}$$

This equation is either proven in a manner similar to the proof of Green's theorem in the plane, but extended to three dimensions, or it is heuristically derived by using an alternate definition of the divergence in terms of limits of ratios of integrals:

$$\nabla \cdot \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint\!\!\oint \mathbf{D} \cdot d\mathbf{S}}{\Delta v}. \tag{18}$$

In cartesian tensor notation the divergence theorem is written as

$$\oint\!\!\oint D_i n_i\, dS = \iiint \partial D_i/\partial x_i\, dv \tag{19}$$

where n_i is the outward (unit vector) normal to the surface.

If the derivation of eq.(17) is done in the notation of eq.(19) one sees that there is nothing special about the integrand having only one tensor subscript. In fact, one might as well derive a more general integral theorem which is

$$\oiint A_{ijk...l...mnp} n_q \, dS = \iiint \partial A_{ijk...l...mnp}/\partial x_q \, dv. \tag{20}$$

The general form of eq.(20) may be specialized to what would be a "curl theorem" in ordinary vector notation. Let $A_{ijk...l...mnp}$ be replaced by $\epsilon_{ijk} B_k$ to obtain

$$\oiint \epsilon_{ijk} B_k n_j \, dS = \iiint \partial \epsilon_{ijk} B_k/\partial x_j \, dv \tag{21}$$

which in standard vector notation is

$$\oiint (-\mathbf{B}) \times d\mathbf{S} = \iiint \nabla \times \mathbf{B} \, dv. \tag{22}$$

A "gradient theorem" may be obtained by replacing $A_{ijk...l...mnp}$ with a scalar to yield

$$\oiint f n_j \, dS = \iiint \partial f/\partial x_j \, dv \qquad \longleftrightarrow \qquad \oiint f d\mathbf{S} = \iiint \nabla f \, dv. \tag{23}$$

Other possible applications of this method will find application in special situations.

Appendix A Derivation of fundamental identity between Levi-Civita tensors and Kroneker deltas

First note that from eq.(8)

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = \epsilon_{ijk} \delta_{1i} \delta_{2j} \delta_{3k} = \epsilon_{123} = 1 \tag{24}$$

where we have used the property of contracting a Kroneker delta with the subscripts on the Levi-Civita symbol. Now if we replace the elements of the determinant $|A|$ in eq.(8) by Kroneker deltas as follows:

$$\begin{aligned} a_{11} &= \delta_{l1}, \quad a_{12} = \delta_{l2}, \quad a_{13} = \delta_{l3} \\ a_{21} &= \delta_{m1}, \quad a_{22} = \delta_{m2}, \quad a_{23} = \delta_{m3} \\ a_{31} &= \delta_{n1}, \quad a_{32} = \delta_{m2}, \quad a_{33} = \delta_{n3} \end{aligned}$$

then we have

$$\epsilon_{lmn} = \epsilon_{ijk} \delta_{li} \delta_{mj} \delta_{nk} = \begin{vmatrix} \delta_{l1} & \delta_{l2} & \delta_{l3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \end{vmatrix} \tag{25}$$

Now, the value of a determinant is unchanged when rows and columns are interchanged, that is, when the matrix is transposed before taking the determinant. Also the letters used for the subscripts are immaterial. Hence we can rewrite ϵ_{ijl} as eq.(25) with the matrix transposed and lmn replaced by ijk. Then we can multiply these two determinants together, and use the property that the determinant of a product of two matrices is the product of the determinants of the individual matrices to obtain

$$\epsilon_{lmn}\epsilon_{ijk} = \begin{vmatrix} \delta_{l1} & \delta_{l2} & \delta_{l3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \end{vmatrix} \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix} = \begin{vmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{vmatrix} \tag{26}$$

where we have used the fact that

$$\delta_{l1}\delta_{i1} + \delta_{l2}\delta_{i2} + \delta_{l3}\delta_{i3} = \delta_{ip}\delta_{lp} = \delta_{il}$$

to obtain the upper left element of the product matrix, and so on.

A less formal presentation of the same results can be demonstrated by looking at the following determinant and observing that this makes ϵ_{123} the determinant of the identity matrix, which has the value 1; every interchange of a row changes the sign of the determinant; for two rows equal the determinant is zero.

$$\epsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \tag{27}$$

In the same way, examining the determinant

$$\epsilon_{lmn}\epsilon_{ijk} = \begin{vmatrix} \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \\ \delta_{in} & \delta_{jn} & \delta_{kn} \end{vmatrix} \tag{28}$$

shows that $\epsilon_{123}\epsilon_{123} = 1$ and, again, any interchange of either rows or columns changes the sign, and any two rows or columns identical makes the value zero. Hence the identity of the product of epsilon symbols and the determinant is shown.

To complete the demonstration of eq.(9) one needs to use eq.(28) to form $\epsilon_{ijk}\epsilon_{klm}$, expand the determinant into a sum of terms of products of Kroneker deltas and then do the summation on k:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{klm} &= \begin{vmatrix} \delta_{ik} & \delta_{jk} & \delta_{kk} \\ \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \end{vmatrix} \\ &= \delta_{ik}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jk}\delta_{kl} + \delta_{kk}\delta_{il}\delta_{jm} \\ &\quad -\delta_{im}\delta_{jl}\delta_{kk} - \delta_{il}\delta_{jk}\delta_{km} - \delta_{ik}\delta_{jm}\delta_{kl} \\ &= \delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} + 3\delta_{il}\delta_{jm} \\ &\quad -3\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} - \delta_{il}\delta_{jm} \\ &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \end{aligned} \tag{29}$$

where the summation on index k has been done. Note $\delta_{kk} = 3$.

References

[1] Tullio Levi-Civita, *The Absolute Differential Calculus (Calculus of Tensors)*, edited by Enrico Persico, translated by Marjorie Long, Blackie & Son limited, London, Glasgow, 1929.

[2] I. S. Sokolnikoff, *Tensor Analysis*, John Wiley & Sons, Inc., New York, 1951.

[3] C. Møller, *The Theory of Relativity*, Oxford, London, 1952.

[4] Richard E. Haskell, *Introduction to Vectors and Cartesian Tensors, A Programmed Text for Students of Engineering and Science*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1972.

[5] D. E. Borne and P. C. Kendall, *Vector Analysis and Cartesian Tensors*, 2nd Ed., Thomas Nelson & Sons Ltd., Great Britain, 1977.