Chapter 5

Orthogonal Curvilinear Coordinates

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Syllabus section:
4. Orthogonal curvilinear coordinates; length of line element; grad, div and curl in curvilinear coordinates; spherical and cylindrical polar coordinates as examples.

So far we have only used Cartesian x,y,z coordinates. Sometimes, because of the geometry of a given problem, it is easier to work in some other coordinate system. Here we show how to do this, restricting the generality only by an orthogonality condition.

5.1 Plane Polar Coordinates

In Calculus II and Chapter 2, we met the simple curvilinear coordinates in two dimensions, plane polars, defined by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We can easily invert these relations to get

$$r = \sqrt{x^2+y^2}, \qquad \theta = \arctan(y/x).$$

The Chain Rule enables us to relate partial derivatives with respect to x and y to those with respect to r and θ and vice versa, e.g.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \qquad = \frac{x}{r}\frac{\partial f}{\partial x} + \frac{y}{r}\frac{\partial f}{\partial y}\ . \tag{5.1}$$

In Calculus II, the rule for changing coordinates in integrals is also given. The general rule is that if we change coordinates from x,y to u,v where $x = x(u,\,v)$, $y = y(u,\,v)$, then a $\mathrm{d}x\,\mathrm{d}y$ in an area integral is replaced by the Jacobian determinant

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u\,\mathrm{d}v\ .$$

If we define $\mathbf{r} = (x(u,v), y(u,v), 0)$, differentiate w.r.t. u,v and take the cross-product, we will see that the above is equal to

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v \qquad ,$$

as we derived in section 4.2.

For plane polar coordinates, replacing u, v with r, θ and calcuating the determinant above gives $\mathrm{d}S = r\,\mathrm{d}r\,\mathrm{d}\theta$; this can also be shown geometrically by considering an infinitesimal quadrilateral with corners at $(r,\theta),\ldots,(r+dr,\theta+d\theta)$ and working out the area from a sketch.

Example 5.1. The Gaussian integral (related to the Gaussian distribution in statistics)

Consider the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2.$$

Transforming to polar coordinates gives

$$\int_0^{\infty} re^{-r^2}\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta = [-\tfrac{1}{2}e^{-r^2}]_0^{\infty}[\theta]_0^{2\pi} = \pi$$

and hence (according to Dr. Saha "the most beautiful of all integrals")

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

For later use, we now construct the unit vectors in the directions in which r and θ increase at a point, which we will denote $\mathbf{e}_r$ and $\mathbf{e}_\theta$. These are tangent to the coordinate lines, where a **coordinate line** means a curve on which only one of the coordinates is varying, and the other coordinates are fixed. Coordinate lines are generalizations of lines parallel to the x, y, z axes in Cartesians, but now they won't be straight lines (hence the "**curvilinear**" in the chapter title).

We already know how to find the tangent vectors to coordinate lines, by taking partial derivatives of $\mathbf{r}$ with respect to each of r, θ; then all we have to do is divide those by their lengths to get unit vectors. Thus in plane polars we have

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} \quad ,$$

so a small change dr gives us a change

$$\mathrm{d}\mathbf{r}_r = \frac{\partial\mathbf{r}}{\partial r}\mathrm{d}r = (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\mathrm{d}r, \qquad \left|\frac{\partial\mathbf{r}}{\partial r}\right| = 1 \quad \Rightarrow \quad \mathbf{e}_r = \frac{\partial\mathbf{r}}{\partial r} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

while a small change $d\theta$ gives us

$$\mathrm{d}\mathbf{r}_\theta = \frac{\partial\mathbf{r}}{\partial\theta}\mathrm{d}\theta = (-r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j})\mathrm{d}\theta, \qquad |\frac{\partial\mathbf{r}}{\partial\theta}| = r \quad \Rightarrow \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

So a general small displacement becomes

$$\delta\mathbf{r} = \mathbf{e}_r\,\delta r + r\,\mathbf{e}_\theta\,\delta\theta \quad .$$

We will see the value of this later on; we are next going to consider three-dimensional versions of polar coordinates: there are two common versions, firstly cylindrical polars and then spherical polars.

5.2 Cylindrical Polar Coordinates

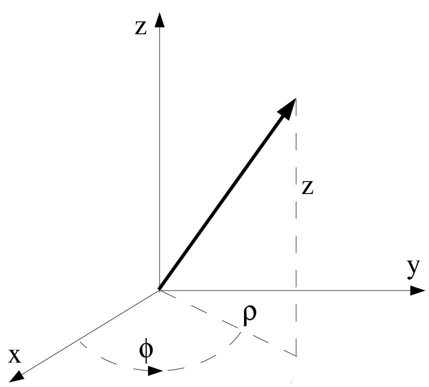
For cylindrical polars, we turn the plane polars in the x, y plane into three-dimensional coordinates by simply using z as the third coordinate (see Fig. 5.1). To avoid confusion with other coordinate systems, we shall for clarity [1] rename r as ρ and θ as ϕ , but beware that in other courses, books, and applications of these ideas, r and θ will still be used. Thus we have

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z\ ,$$

or

$$\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\mathbf{k} \quad ,$$

and quantities in any plane $z =$constant will be as in plane polars. The figure 5.1 shows coordinate lines for each of ρ, ϕ and z; here the coordinate line for ρ is a line of varying ρ and constant ϕ,z; and likewise for the other two. Note that the coordinate lines for ρ, z are straight lines, while the ϕ line is a circle around the z axis. Thomas's Fig. 15.37 shows a nice diagram of surfaces on which one of the coordinates is constant: the constant-ρ surface is a cylinder whose axis is the $z-$axis, while surfaces of constant ϕ or constant z are planes.



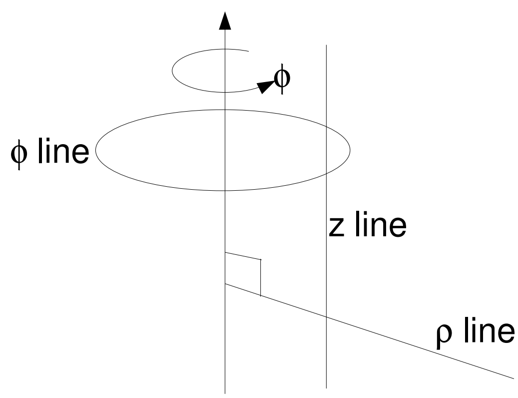


Figure 5.1: Cylindrical polar coordinates relative to Cartesian, and with sample ρ- and ϕ-curves shown.

The fact that constant ρ gives a cylinder gives the name cylindrical polars: these coordinates are natural ones to use whenever there is a problem involving cylindrical geometry or symmetry (for example, doing a surface integration over a cylinder, or in physics calculating a magnetic field around a straight wire).

To get partial derivatives in curvilinear coordinates we again use the chain rule (5.1), but now with three terms on the right. Taking the plane polar results, changing variable names and appending $\mathbf{e}_z = \mathbf{k}$, the unit vectors along the coordinate lines are

$$\mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}\ , \quad \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}\ , \quad \mathbf{e}_z = \mathbf{k}$$

respectively. We can write this in matrix form as

$$\begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}. \qquad (5.2)$$

[1] Unfortunately, for the same reasons of clarity, Thomas adopts the alternative solution of renaming two of the spherical polar coordinates. To avoid confusion with past years' exam papers I have kept to the choice used there, which is also the one used in most books. Thomas chooses (ρ, ϕ, θ) for the usual (r, θ, ϕ). The swap of θ and ϕ is particularly likely to be confusing.

It is easy to see from the above that the dot-product of any two $\mathbf{e}$'s gives 1 (if they are the same) or 0 (for any two different ones), like the rules for $\mathbf{i},\mathbf{j},\mathbf{k}$. This implies that the three $\mathbf{e}$'s are an **orthogonal triple** of unit vectors, and also implies geometrically that the cross-product of any two different $\mathbf{e}$'s will be $\pm$ the third one.

We can also express this property in matrix notation: the 3×3 matrix above, call it $\mathbf{R}$, is a rotation matrix, i.e. one such that $\mathbf{R}^{-1} = \mathbf{R}^T$, where the $\mathbf{R}^T$ denotes transpose. This comes about because the dot-product of any two $\mathbf{e}$'s is given by one element of the matrix $\mathbf{R}\mathbf{R}^T$, and the $\mathbf{e}$'s are an orthogonal triple if and only if $\mathbf{R}\mathbf{R}^T = \mathbf{I}$, the identity matrix. [2] Also note that if we want $\mathbf{i},\mathbf{j},\mathbf{k}$ in terms of the $\mathbf{e}$'s, we can just multiply Eq. 5.2 by $\mathbf{R}^{-1} = \mathbf{R}^T$.

The lengths of $\partial\mathbf{r}/\partial\rho$, $\partial\mathbf{r}/\partial\phi$ and $\partial\mathbf{r}/\partial z$ are respectively 1, ρ and 1; we can use these together with the $\mathbf{e}$'s to find infinitesimal area elements: e.g. taking a surface $\rho =$constant (a cylinder), we can treat this as a 2-parameter surface with ϕ, z as the parameters, so the vector area element for small changes $d\phi, dz$ is given by

$$\begin{aligned}
\mathrm{d}\mathbf{S} &= \frac{\partial\mathbf{r}}{\partial\phi}\times\frac{\partial\mathbf{r}}{\partial z}\,\mathrm{d}\phi\,\mathrm{dz} \\
&= \rho\mathbf{e}_\phi\times\mathbf{e}_z\,\mathrm{d}\phi\,\mathrm{dz} \\
&= \rho\ \mathbf{e}_\rho\mathrm{d}\phi\,\mathrm{dz}\ ;
\end{aligned}$$

this will be useful when doing surface integrals over a cylinder. (As usual, there is a potentially ambiguous choice of sign with vector areas, due to the sign-flip in changing order of a cross product; take care with this, e.g. when doing a problem check that your vector area matches the desired direction).

When doing volume integrals, we may need the volume element which is

$$\mathrm{d}V = \rho\ \mathrm{d}\rho\,\mathrm{d}\phi\mathrm{dz}$$

from the scalar triple product.

5.3 Spherical Polar Coordinates

These are coordinates (r,θ,ϕ), where r measures distance from the origin, θ measures angle from some chosen axis, called the **polar axis**, and ϕ measures angle around that axis (see Fig 5.2.) To relate them to Cartesian coordinates we usually assume that the z-axis is the polar axis. Then, let P be our chosen point at (r,θ,ϕ), and drop a perpendicular from P to the $z-$axis meeting it at Q. The line OP is at angle θ to the positive z-axis, so clearly $OQ = z = r\cos\theta$ and PQ = $r\sin\theta$. Dropping another perpendicular from P to the xy plane, we get a point in the xy plane at distance $r\sin\theta$ from the origin; then inserting $\rho = r\sin\theta$ into the cylindrical polars in Sec. 5.2 gives us:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

or, as a position vector

$$\mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

Here the ϕ is the same as that of cylindrical polars, which explains why we chose the same letter. The inverse of these relations is

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right), \quad \phi = \arctan\left(\frac{y}{x}\right).$$

[2] Rotation matrices are "special" because they preserve lengths and angles; e.g. if we take two vectors $\mathbf{a}$, $\mathbf{b}$, write them as column vectors, then their scalar product in matrix notation is $\mathbf{a}^T\mathbf{b}$. The two vectors rotated by matrix $\mathbf{R}$ are $\mathbf{Ra}$ and $\mathbf{Rb}$. To conserve scalar product, we must have $(\mathbf{R}\mathbf{a})^T(\mathbf{R}\mathbf{b}) = \mathbf{a}^T\mathbf{b}$, and using the transpose rule this becomes $\mathbf{a}^T\mathbf{R}^T\mathbf{Rb} = \mathbf{a}^T\mathbf{b}$. For this to apply for *any* two $\mathbf{a},\mathbf{b}$ we must have $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, the identity matrix.

Coordinate lines of r, (i.e. lines of constant θ and ϕ), are straight radial lines from the origin; coordinate

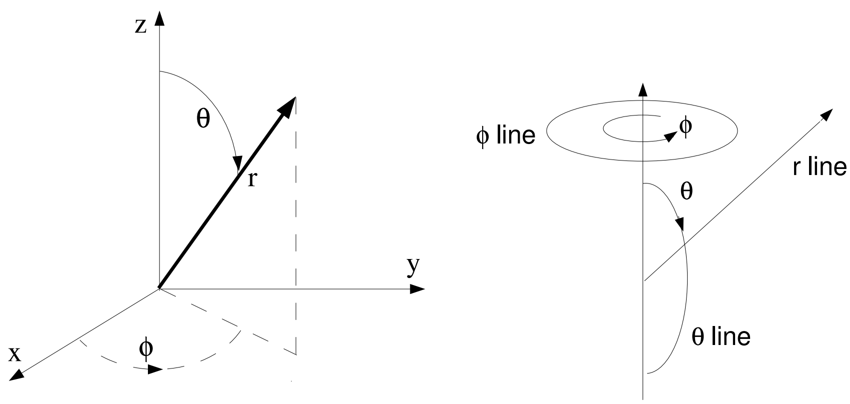


Figure 5.2: Spherical polar coordinates relative to Cartesian, and with sample r-, $\theta-$ and ϕ-curves shown.

lines of θ (constant r and ϕ) are **meridional** semicircles, i.e. semicircles centred at the origin and in a plane containing the polar axis; and coordinate lines of ϕ (constant r and θ) are **latitudinal** circles, i.e. circles centred at a point on the polar axis and in a plane perpendicular to it. Note however that while r runs from 0 to ∞ (like the r of plane polars and ρ of cylindrical polars) and ϕ runs from 0 to 2π (like the θ of plane polars), θ only runs **from 0 to** π, since for any point P the angle between OP and the z-axis won't exceed 180 degrees = π radians.

The coordinate lines of θ are strictly semi-circles, rather than circles. To make a circle we have to take the coordinate lines of θ for two different ϕ, say ϕ_0 and $\phi_0+\pi$. Thomas's Fig. 15.42 shows a nice diagram of surfaces on which one of the coordinates is constant.

You should beware of the fact that some authors, including Thomas, use different notation, in particular swapping the meanings of θ and ϕ in the definition of spherical polars. We shall consistently use the above notation for spherical polar coordinates, which is the most common one, throughout this course.

Note that these again generalize the plane polar coordinates, but this time the polars r, θ are in planes containing the z (or polar) axis, rather than in planes perpendicular to it. The spherical polar coordinates are of course the natural ones to use when we have a spherical geometry, or part of a sphere.

Now we construct the **e** vectors as before: taking partial derivatives of **r** above with respect to each of the coordinates in turn, we get

$$\begin{aligned}
\partial\mathbf{r}/\partial r &= \sin\theta\cos\phi\,\mathbf{i}+\sin\theta\sin\phi\,\mathbf{j}+\cos\theta\,\mathbf{k},\\
\partial\mathbf{r}/\partial\theta &= r\cos\theta\cos\phi\,\mathbf{i}+r\cos\theta\sin\phi\,\mathbf{j}-r\sin\theta\,\mathbf{k}\\
\partial\mathbf{r}/\partial\phi &= -r\sin\theta\sin\phi\,\mathbf{i}+r\sin\theta\cos\phi\,\mathbf{j}.
\end{aligned}$$

The lengths of these, by simple applications of $\cos^2\phi+\sin^2\phi=1$, are respectively 1, r, and $r\sin\theta$. Dividing these derivatives by their lengths gives us the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ tangent to the coordinate lines, which we can write as

$$\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\\ \mathbf{e}_\phi\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix}. \qquad (5.3)$$

It is straightforward to show that again the dot-product of any two **e**'s is 1 (if they are the same) or 0 (if different); therefore the cross-product of any two **e**'s is $\pm$ the third one and the matrix above is again a rotation matrix.

It is also worth noting that $\mathbf{e}_r = \mathbf{r}/r$, as expected by symmetry since $\mathbf{e}_r$ is a unit vector pointing away from the origin at point $\mathbf{r}$.

For doing integrals later on, the volume element is given by the scalar triple product

$$dV = (\partial \mathbf{r}/\partial r) \times (\partial \mathbf{r}/\partial \theta).(\partial \mathbf{r}/\partial \phi)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \quad .$$

The infinitesimal area element on a sphere (i.e. a surface of constant r) is given by

$$d\mathbf{S} = (\partial \mathbf{r}/\partial \theta) \times (\partial \mathbf{r}/\partial \phi)\, \mathrm{d}\theta\, \mathrm{d}\phi \quad = r^2 \sin\theta\, \mathbf{e}_r\, \mathrm{d}\theta \mathrm{d}\phi \qquad .$$

Similar results hold for surfaces of constant ϕ and of constant θ, but are not so common in practice; note that the above area element on a sphere turns up in many examples and exam questions, and is well worth memorising.

Example 5.2. "Earth polar coordinates"

To define spherical polars on the Earth, let the polar axis be the Earth's rotation axis, with z increasing to the North, let the equator define the x,y plane, and let the prime meridian (the one through Greenwich) be $\phi = 0$. Then any point on the Earth's surface can be referred to by the spherical polar angles (θ, ϕ). In navigation people use latitude and logitude. Longitude is measured East or West from the prime meridian and is in the range $(0, 180^\circ)$ so to get ϕ for a place with Westerly longitude we just subtract from $2\pi = 360^\circ$. Latitude is defined to be 0 at the equator (whereas $\theta = 90^\circ = \pi/2$ there). Given a latitude, we need to subtract it from 90° if it is North and add it to 90° if it is South.

For example Buenos Aires, which has latitude $34^\circ 36'$S, and longitude $58^\circ 22'$W, will have Earth polar coordinates $\theta = 125^\circ, \phi = 302^\circ$ to the nearest degree.

5.4 Some applications of these polar coordinates

Using polar (or cylindrical) coordinates the area within a circle of radius R, $\int_0^R \int_0^{2\pi} r\, \mathrm{d}\phi\, \mathrm{d}r$, comes out immediately as πR^2.

Using spherical polar coordinates the volume of a sphere of radius R is

$$\int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\theta\; \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}r$$

which evaluates to $\frac{4}{3}\pi R^3$. (Remember that for a full sphere, the ranges of integration are $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$).

Example 5.3. Area of a cone:

Consider the conical surface $\theta = \theta_1$ cut in a sphere of radius s. The area is given by integrating

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^s \sin\theta_1 r\; \mathrm{d}r = \pi s^2 \sin\theta_1 .$$

Here s is the slant height of the cone. The cone's base (say b) will be $s \sin\theta_1$. Hence we can express the sloping area of a cone neatly as $\pi s b$.

Example 5.4. We now reconsider Example 4.5.
Find the flux of the field $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant with normal taken in the direction away from the origin.

Because of the geometry of the surface, it is easiest to work in spherical polar coordinates (r,θ,ϕ), so the sphere has $r=a$. The unit normal $\mathbf{n}$ to the sphere that points away from the origin is just $\mathbf{e}_r$, the outward radial vector of unit length. Now

$$\mathbf{F}.\mathbf{e}_r = z\mathbf{k}\cdot\mathbf{e}_r = z\cos\theta = r\cos^2\theta\ .$$

using 5.3 to evaluate $\mathbf{k}.\mathbf{e}_r=\cos\theta$. An area element on the surface of a sphere of radius r is $(r\mathrm{d}\theta)(r\sin\theta\,\mathrm{d}\phi) = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. For our given sphere $r=a$, so

$$\begin{aligned}\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int_0^{\pi/2}\int_0^{\pi/2} a\cos^2\theta\, a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= a^3\int_0^{\pi/2}\int_0^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi\\ &= \frac{\pi}{2}a^3\left[\frac{-1}{3}\cos^3\theta\right]_0^{\pi/2}\\ &= \frac{\pi}{6}a^3\ .\end{aligned}$$

Note that the integrand didn't depend on ϕ, so we just replaced the $d\phi$ integral with a multiplication by the range, here $(\pi/2-0)$. This is a common short-cut to note.

Example 5.5. Cutting an apple

In his book, Matthews poses a good problem for illustrating integration using curved coordinates: "A cylindrical apple corer of radius a cuts through a spherical apple of radius b. How much of the apple does it remove?"

We can reformulate the problem slightly, without losing generality, by letting the radius of the apple equal unity and introducing $\sin\theta_1 = a/b$ (i.e. we scale the problem by b). In our restated problem the corer cuts through the peel at $\theta=\theta_1$ and $\theta=\frac{1}{2}\pi-\theta_1$ in spherical polars, i.e. in cylindrical polars at

$$\rho=\sin\theta_1,\quad z=\cos\theta_1,$$

and, of course, at $z=-\cos\theta_1$.

We can now complete the solution of this problem in (at least) four different ways: three of these are relegated to an appendix, not given in lectures.[3]

The first way is to integrate over z and then ρ

$$4\pi\int_0^{\sin\phi_1}\rho\,d\rho\int_0^{\sqrt{1-\rho^2}}dz = 4\pi\int_0^{\sin\phi_1}\rho(1-\rho^2)^{\frac{1}{2}}\,d\rho = \frac{4\pi}{3}(1-\cos^3\phi_1).$$

5.5 General Orthogonal Curvilinear Coordinates

The two sets of polar coordinates above have a feature in common: the three sets of coordinate lines are orthogonal to one another at all points, because their tangent vectors and corresponding unit vectors $\mathbf{e}$'s are orthogonal. (This is where the **orthogonal** in the chapter title comes from).

[3] I give only the key steps. Some algebraic filling-in is needed. In each version we can shorten the calculations by replacing the ϕ integration with multiplication by 2π (since the integrand doesn't depend on ϕ) , and also doing the integrals only for $z\geq 0$, and then doubling using symmetry.

General orthogonal coordinates are coordinates for which these properties are true, i.e. the coordinate lines are always mutually perpendicular **at a given point**, though they are generally curved. In general, coordinates need not be orthogonal. However, we shall be concerned only with orthogonal curvilinear coordinates. Cylindrical polars and spherical polars are the only non-Cartesian coordinate systems in which you will be expected to perform explicit calculations in this course, apart from simple substitutions into the general formulae.

Suppose (u_1, u_2, u_3) are a general set of coordinates, defined by some given function $\mathbf{r}(u_1, u_2, u_3)$. As before, we calculate $\partial\mathbf{r}/\partial u_1$ which is the tangent vector to a u_1 line (varying u_1, constant u_2, u_3). Next we define the arc-length h_1 and unit vector $\mathbf{e}_1$ as

$$h_1 = \left|\frac{\partial\mathbf{r}}{\partial u_1}\right|, \qquad \mathbf{e}_1 = \frac{\partial\mathbf{r}}{\partial u_1}/h_1$$

therefore

$$\frac{\partial\mathbf{r}}{\partial u_1} \equiv h_1\mathbf{e}_1 \quad .$$

It is easy to calculate that

$$h_1^2 = \left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2 .$$

Likewise differentiating $\mathbf{r}$ by u_2, u_3, we define two more unit vectors $\mathbf{e}_2$, $\mathbf{e}_3$, along the coordinate lines of u_2 and u_3, and associated arc-length parameters h_2 and h_3. This is useful for several reasons: firstly, $\mathbf{e}_1$ tells us in which direction $\mathbf{r}$ moves with a small change in u_1, while $h_1\,du_1$ is the distance moved along $\mathbf{e}_1$, and likewise for changes du_2, du_3.

We define a coordinate system to be **orthogonal** iff $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are mutually orthogonal everywhere:

$$\text{Coordinates } (u_1, u_2, u_3) \text{ are orthogonal} \Leftrightarrow \frac{\partial\mathbf{r}}{\partial u_1}.\frac{\partial\mathbf{r}}{\partial u_2} = \frac{\partial\mathbf{r}}{\partial u_2}.\frac{\partial\mathbf{r}}{\partial u_3} = \frac{\partial\mathbf{r}}{\partial u_3}.\frac{\partial\mathbf{r}}{\partial u_1} = 0$$

For orthogonal coordinates, a general small change $(\mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}u_3)$ in the coordinates means a displacement

$$d\mathbf{r} \quad = \quad h_1\mathrm{d}u_1\mathbf{e}_1 + h_2\mathrm{d}u_2\mathbf{e}_2 + h_3\mathrm{d}u_3\mathbf{e}_3 \quad , \tag{5.4}$$

which corresponds to a distance

$$\left(h_1^2\mathrm{d}u_1^2 + h_2^2\mathrm{d}u_2^2 + h_3^2\mathrm{d}u_3^2\right)^{1/2} .$$

Also, for orthogonal coordinates the dot and cross products of any two **e**'s will obey the same rules we met before: therefore the matrix $\mathbf{R}$ relating $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ will be a rotation matrix (from above) and have the property that $\mathbf{R}^T = \mathbf{R}^{-1}$.

Cartesian coordinates are of course a special simple case of orthogonal curvilinear coordinates, in which all the coordinate lines are straight lines and all of $h_1 = h_2 = h_3 = 1$.

Sometimes it is convenient to replace the 1,2,3 with the letters of the coordinates, e.g. in cylindrical polar coordinates, we wrote $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$. There we already found $h_\rho = 1$ and $h_z = 1$, but $h_\phi = \rho$, so a change $\mathrm{d}\phi$ corresponds to moving a distance $\rho\mathrm{d}\phi$ along a circle around the $z-$axis.

In spherical polar coordinates, $h_r = 1$ again, and $h_\theta = r$. A change $\mathrm{d}\phi$ in ϕ corresponds to moving a distance $r\sin\theta\,\mathrm{d}\phi$ (because $r\sin\theta$ is the radius of the particular latitudinal circle around the $z-$axis), so $h_\phi = r\sin\theta$.

One reason that orthogonal coordinates are so useful is that in any orthogonal coordinate system (u_1, u_2, u_3), small displacements along u_1 and u_2 define small rectangles, while small displacements along u_1, u_2, u_3 define small cuboids. In other words, $h_1\,h_2\,du_1\,du_2$ is an area element normal to $\mathbf{e}_3$ on a surface of constant u_3, and $h_1\,h_2\,h_3\,du_1\,du_2\,du_3$ is a volume element.

5.6 Vector fields and vector algebra in curvilinear coordinates

Scalar fields can of course be expressed in (orthogonal) curvilinear coordinates: they are simply written as functions $f(u_1, u_2, u_3)$ or for brevity $f(u_i)$.

As you will know from Linear Algebra, vectors can be expressed using any basis of the vector space concerned. The same is true, at each point, of vector fields. Up to now we have always chosen $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as our basis vectors: however, when using curvilinear coordinates we will normally use the orthogonal unit vectors along the coordinate lines as our basis vectors, and write

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3.$$

For clarity, we can use the coordinate names instead of 1,2,3 as subscripts for the three components. Thus we may write

$$\begin{aligned}\mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \\ &= F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z \\ &= F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi.\end{aligned}$$

to express the same vector in Cartesian, cylindrical polar and spherical polar coordinates (of course $\mathbf{e}_x = \mathbf{i}$ and so on in Cartesians). Note that the **same** vector $\mathbf{F}$ will have **different** components depending on our choice of basis vectors: suppose we are given an $\mathbf{F}$ with defined F_x, F_y, F_z above, but we want to find F_r, F_θ, F_ϕ, then we need to use the matrix as in Eq. 5.3 to express $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in terms of the $\mathbf{e}$'s, multiply out and collect into one term in each $\mathbf{e}$. (This effectively turns into a matrix multiplication).

In any orthogonal coordinate system, the scalar (dot) and vector (cross) products work just as in Cartesian coordinates:

$$\mathbf{w}.\mathbf{v} = w_1v_1 + w_2v_2 + w_3v_3 \tag{5.5}$$

and

$$\mathbf{w}\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ w_1 & w_3 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad , \tag{5.6}$$

but note this only works if the vectors are defined **at the same point** , such as a dot product $\mathbf{F}\cdot d\mathbf{r}$ or $\mathbf{F}\cdot d\mathbf{S}$ in a line or surface integral. We **cannot** use these for two position vectors at widely separate points, because the $\mathbf{e}$'s vary with position.

Vector differentiation is more complicated, because the unit vectors are no longer constant: when we differentiated a vector in Cartesians

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

we just differentiated the components (F_1, F_2, F_3) because the unit vectors are constant; but in general coordinates the $\mathbf{e}$'s depend on position, so we have to use the product rule and differentiate the $\mathbf{e}$ vectors as well as the components F_i.

Differentiation of these vectors with respect to a variable other than position (like the derivatives in Section 3.1) is straightforward. For example if position $\mathbf{r}$ depends on time, and is given in cylindrical polars so $\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$, we just use the product rule to get the time derivative

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\mathbf{e}}_\rho + \dot{z}\mathbf{e}_z + z\dot{\mathbf{e}}_z \ .$$

(where the over-dots are shorthand for time derivative, as is common). Then since $\mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$ from (5.2),

$$\dot{\mathbf{e}}_\rho = \dot{\phi}(-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) = \dot{\phi}\mathbf{e}_\phi \ .$$

Similarly $\dot{\mathbf{e}}_z = \mathbf{0}$. Substituting into the previous result, we get

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z$$

for a velocity in cylindrical polar coordinates.

When differentiating scalar or vector fields with respect to position, the key operations are always grad of a scalar, and div and curl of a vector field (this is because these are the only combinations that behave "sensibly" after rotations). In the next sections, we will show how to calculate the grad, div and curl operators in general orthogonal coordinates; then we apply those general formulae to the most common cases of cylindrical polars and spherical polars.

5.7 The Gradient Operator in curvilinear coordinates

To calculate the gradient of a scalar field $V(u_1, u_2, u_3)$ in orthogonal curvilinear coordinates (u_1, u_2, u_3), we go back to the definition

$$\mathrm{d}V = \nabla V \cdot \mathrm{d}\mathbf{r} \qquad . \tag{$*$}$$

for the change dV caused by an infinitesimal position change $d\mathbf{r}$.

(**Note:** here dV is the infinitesimal change in scalar field V resulting from a small change $d\mathbf{r}$; it is not a volume element.)

We define $\nabla V \equiv (\nabla V)_1\mathbf{e}_1 + (\nabla V)_2\mathbf{e}_2 + (\nabla V)_3\mathbf{e}_3$, and we want to find the three components $(\nabla V)_1$ etc.

From the definitions of the unit vectors previously, we have $\mathrm{d}\mathbf{r} = \mathbf{e}_1 h_1 \mathrm{d}u_1 + \mathbf{e}_2 h_2 \mathrm{d}u_2 + \mathbf{e}_3 h_3 \mathrm{d}u_3$, so the right-hand side of $(*)$ becomes

$$\begin{aligned} ((\nabla V)_1\mathbf{e}_1 + (\nabla V)_2\mathbf{e}_2 + (\nabla V)_3\mathbf{e}_3) \cdot \quad & (\mathbf{e}_1 h_1 \mathrm{d}u_1 + \mathbf{e}_2 h_2 \mathrm{d}u_2 + \mathbf{e}_3 h_3 \mathrm{d}u_3) \\ & = (\nabla V)_1 h_1 \mathrm{d}u_1 + (\nabla V)_2 h_2 \mathrm{d}u_2 + (\nabla V)_3 h_3 \mathrm{d}u_3 \end{aligned}$$

using the orthogonality of the **e**'s.

Now turning to the left-hand side of of $(*)$, using Taylor's theorem (in 3 dimensions), and discarding terms of second and higher derivatives, we get

$$dV = \frac{\partial V}{\partial u_1}\mathrm{d}u_1 + \frac{\partial V}{\partial u_2}\mathrm{d}u_2 + \frac{\partial V}{\partial u_3}\mathrm{d}u_3$$

These two expressions above must be equal for **any** arbitrary changes $\mathrm{d}u_1$, $\mathrm{d}u_2$ and $\mathrm{d}u_3$. Hence we must have

$$(\nabla V)_1 h_1 = \frac{\partial V}{\partial u_1} \ ; \quad (\nabla V)_2 h_2 = \frac{\partial V}{\partial u_2} \ ; \quad (\nabla V)_3 h_3 = \frac{\partial V}{\partial u_3} \quad .$$

Dividing by the h's and substituting back into the original definition, in orthogonal curvilinear coordinates we have

$$\nabla V \quad = \quad \frac{1}{h_1}\frac{\partial V}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\mathbf{e}_3 \quad . \tag{5.7}$$

Clearly in Cartesian coordinates, we have $u_1 = x$, $\mathbf{e}_1 = \mathbf{i}$ etc and all three h's are 1, so this simplifies to the well-known formula from Chapter 1.

For a geometrical explanation, the $1/h_i$ terms take care of the arc-length effects, i.e. how far $\mathbf{r}$ moves for a small change in each coordinate. So the 1-component of ∇V represents the change dV per small **distance**

ds in the direction $\mathbf{e}_1$; but, moving a distance ds in direction $\mathbf{e}_1$ requires a change $\delta u_1 = ds/h_1$ in coordinate u_1; therefore the $1/h_i$ terms appear in grad V above.

Example 5.6. What is ∇V in spherical polar coordinates ? Evaluate ∇V where $V = r\sin\theta\cos\phi$.

In spherical polars, $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$. Putting those into 5.7 we have

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{e}_\phi \quad .$$

For the given V, $\partial V/\partial r = \sin\theta\cos\phi$, $\partial V/\partial\theta = r\cos\theta\cos\phi$ and $\partial V/\partial\phi = -r\sin\theta\sin\phi$. Hence, using the result above,

$$\nabla V = \sin\theta\cos\phi\,\mathbf{e}_r + \cos\theta\cos\phi\,\mathbf{e}_\theta - \sin\phi\,\mathbf{e}_\phi .$$

(In this case we can observe that $V = x$ and $\nabla V = \mathbf{i}$, using the matrix from Eq. 5.3, so this example is a lot easier in Cartesians; however, many problems involving circular or spherical symmetry do get easier in polar coordinates).

Exercise 5.1. What is ∇V in cylindrical polar coordinates (ρ, ϕ, z) ? □

Exercise 5.2. Let (r, θ, ϕ) be spherical polar coordinates. Evaluate ∇f where

$$\text{(a) } f = \phi\,; \quad \text{(b) } f = \theta\,; \quad \text{(c) } f = (r^n \sin m\theta)\,.$$

□

5.8 The Divergence Operator in curvilinear coordinates

Next we want to compute $\nabla \cdot \mathbf{F}$ in orthogonal curvilinear coordinates. Although we could directly calculate the divergence in any coordinates, using the Cartesian definition, the matrix relating basis unit vectors, and the chain rule, the results can be found with less effort from the Divergence Theorem. The Divergence Theorem is true in all coordinates (since it equates scalars, whose value must be independent of the coordinates). Thus

$$\int_V \nabla \cdot \mathbf{F}\, \mathrm{d}V = \int_{\mathscr{S}} \mathbf{F}.\mathrm{d}\mathbf{S}\ ,$$

where $\mathscr{S}$ is the closed surface enclosing volume V.

Now, we apply this to an infinitesimal "cuboid" with one corner at (u_1, u_2, u_3) and edges corresponding to changes δu_1, δu_2, δu_3 in each coordinate; so this has eight corners at $(u_1, u_2, u_3), (u_1 + \delta u_1, u_2, u_3), \ldots (u_1 + \delta u_1,\, u_2 + \delta u_2,\, u_3 + \delta u_3)$. From before, the volume of the cuboid is $\delta V = (h_1\delta u_1)(h_2\delta u_2)(h_3\delta u_3)$. For a sufficiently small volume, we can approximate $\nabla \cdot \mathbf{F}$ as constant across δV, so the left-hand side becomes

$$(\nabla \cdot \mathbf{F})\delta V = (\nabla \cdot \mathbf{F})(h_1 h_2 h_3 \delta u_1 \delta u_2 \delta u_3)\ .$$

Next we consider the right-hand side of the Divergence Theorem: we need to take the surface integral over the six faces of our cuboid, and add results. First consider the integral of $\mathbf{F}.\mathbf{n}$ over the face of the cuboid where the first coordinate has value $u_1 + \delta u_1$. This face is a rectangle with unit normal $+\mathbf{e}_1$ and area $(h_2\delta u_2)(h_3\delta u_3)$, so the surface integral is approximately

$$(h_2 h_3 \delta u_2 \delta u_3 F_1)_{u_1+\delta u_1}\ ,$$

where the subscript shows it is evaluated at $u_1 + \delta u_1$. On the opposite face at u_1 we have unit normal $-\mathbf{e}_1$ (pointing outwards i.e. away from the first face), so the surface integral gives us

$$-(h_2 h_3 \delta u_2 \delta u_3 F_1)_{u_1} \ .$$

Repeating the above for the other four faces we get symmetrical results; finally summing the six terms and then taking the limit as $\delta V \to 0$, we obtain

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \lim_{\delta u_1, \delta u_2, \delta u_3 \to 0} \frac{1}{\delta V} \Big[\quad (h_2 \delta u_2 h_3 \delta u_3 F_1)_{u_1+\delta u_1} - (h_2 \delta u_2 h_3 \delta u_3 F_1)_{u_1} \\ &\qquad + (h_3 \delta u_3 h_1 \delta u_1 F_2)_{u_2+\delta u_2} - (h_3 \delta u_3 h_1 \delta u_1 F_2)_{u_2} \\ &\qquad + (h_1 \delta u_1 h_2 \delta u_2 F_3)_{u_3+\delta u_3} - (h_1 \delta u_1 h_2 \delta u_2 F_3)_{u_3} \Big] \ . \end{aligned}$$

Though each pair of brackets looks the same, this is not zero because the h's and F's are different on opposite faces of the cuboid; the first two terms give us δu_1 times the partial derivative $\partial / \partial u_1$ of the bracket, and so on for the next pairs, so this gets us the result

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (h_2 h_3 F_1)}{\partial u_1} + \frac{\partial (h_3 h_1 F_2)}{\partial u_2} + \frac{\partial (h_1 h_2 F_3)}{\partial u_3} \right] . \tag{5.8}$$

Note: In this last step, we have taken some δu's outside the brackets and cancelled them with the ones in δV, but we must leave the h's **inside** the differentiation since the h's generally vary with position. This comes about because our "cuboid" may be slightly "tapering", so the areas of opposite faces are not exactly equal; and differentiating the h_i's takes care of that.

Example 5.7. What is $\nabla \cdot \mathbf{F}$ in cylindrical polar coordinates, where $\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z$?

In cylindrical polars, $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$. Hence

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \left[\frac{\partial (\rho F_\rho)}{\partial \rho} + \frac{\partial F_\phi}{\partial \phi} + \frac{\partial (\rho F_z)}{\partial z} \right] .$$

Note that we can apply the product rule, and since $\partial \rho / \partial z = 0$, $\partial \rho / \partial \rho = 1$ we get

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} F_\rho + \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} .$$

Note: It is important to note that an F_ρ term has appeared here, which is not a derivative of F. This has appeared because the coordinate lines for ρ have a "built in divergence", they all radiate outwards from the z-axis, so a field with constant F_ρ has a positive divergence term due to this.

As a further example we can note that in cylindrical polars, $\mathbf{r} = \rho \mathbf{e}_\rho + 0 \mathbf{e}_\phi + z \mathbf{e}_z$. Plugging in components $(\rho, 0, z)$ to the above, we get

$$\nabla \cdot \mathbf{r} = 1 + 1 + 0 + 1 = 3$$

which agrees with the result in Cartesians, as it must.

(If we had just taken $\partial \rho / \partial \rho + \partial z / \partial z$ we would have got $\nabla \cdot \mathbf{r} = 2$; clearly wrong) .

Example 5.8. What is $\nabla \cdot \mathbf{F}$ in spherical polar coordinates, where $\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi$?

In spherical polars, $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1$, $h_2 = r$, $h_3 = r \sin\theta$. Hence

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2 \sin\theta} \left[\frac{\partial (r^2 \sin\theta F_r)}{\partial r} + \frac{\partial (r \sin\theta F_\theta)}{\partial \theta} + \frac{\partial (r F_\phi)}{\partial \phi} \right] .$$

5.9 The Curl Operator in curvilinear coordinates

Finally we want curl: as before we have curvilinear coordinates (u_1, u_2, u_3), and a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$; we want to calculate

$$\nabla \times \mathbf{F} \equiv (\nabla \times \mathbf{F})_1\mathbf{e}_1 + (\nabla \times \mathbf{F})_2\mathbf{e}_2 + (\nabla \times \mathbf{F})_3\mathbf{e}_3 \;,$$

so we want the 1,2,3 components of the above.

In analogy with the previous section, we use Stokes's theorem to provide a coordinate-independent definition of $\nabla \times \mathbf{F}$:

$$\int_{\mathscr{S}} (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathscr{S}$ is a surface spanning the closed curve $\mathscr{C}$.

To calculate the 1-component $(\nabla \times \mathbf{F})_1$, consider a planar curve around a small "rectangle" on a surface of constant u_1, with sides given by small changes δu_2 and δu_3. From previous results, the vector area of this rectangle $d\mathbf{S} = h_2\delta u_2 h_3\delta u_3\,\mathbf{e}_1$; now taking $(\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, the 2 and 3 components of $\nabla \times \mathbf{F}$ disappear so the LHS of Stokes's theorem is approximately

$$(\nabla \times \mathbf{F})_1 h_2\delta u_2 h_3\delta u_3 \quad .$$

Now looking at the RHS of Stokes's theorem, the line integral around the edge of the same rectangle is given by adding the line integrals along the four sides: this is approximately

$$(h_2\delta u_2 F_2)_{u_3} + (h_3\delta u_3 F_3)_{u_2+\delta u_2} - (h_2\delta u_2 F_2)_{u_3+\delta u_3} - (h_3\delta u_3 F_3)_{u_2} \;,$$

where the subscripts denote that the term is evaluated at that value, and two minus signs appear because opposite sides are traversed in opposite directions around the closed rectangle. Equating the last two expressions, and taking the limit as δu_2, $\delta u_3 \to 0$, we have

$$\begin{aligned}(\nabla \times \mathbf{F})_1 &= \frac{1}{h_2h_3} \lim_{\delta u_2, \delta u_3 \to 0} \left[\frac{(h_3F_3)_{u_2+\delta u_2} - (h_3F_3)_{u_2}}{\delta u_2} - \frac{(h_2F_2)_{u_3+\delta u_3} - (h_2F_2)_{u_3}}{\delta u_3} \right] \\ &= \frac{1}{h_2h_3} \left(\frac{\partial(h_3F_3)}{\partial u_2} - \frac{\partial(h_2F_2)}{\partial u_3} \right).\end{aligned}$$

This is just the 1-component of $\nabla \times \mathbf{F}$. To get the 2- and 3- components, we just repeat all the above for two more small rectangles in surfaces of constant u_2, u_3 respectively; this looks the same but cycling the 1/2/3's, and we get

$$\begin{aligned}(\nabla \times \mathbf{F})_2 &= \frac{1}{h_3h_1} \left(\frac{\partial(h_1F_1)}{\partial u_3} - \frac{\partial(h_3F_3)}{\partial u_1} \right), \\ (\nabla \times \mathbf{F})_3 &= \frac{1}{h_1h_2} \left(\frac{\partial(h_2F_2)}{\partial u_1} - \frac{\partial(h_1F_1)}{\partial u_2} \right).\end{aligned}$$

These results can be written in a compact (and more memorable) form as a determinant:

$$\nabla \times \mathbf{F} = \frac{1}{h_1h_2h_3} \begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}. \tag{5.9}$$

Once again, in Cartesian coordinates this simplifies to the well-known expression from Chapter 3.4.

Example 5.9. What is $\nabla \times \mathbf{F}$ in spherical polar coordinates?

In spherical polar coordinates (r,θ,ϕ) we have $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$. Hence, using the determinant form:

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}.$$

or in expanded form

$$\nabla\times\mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(r\sin\theta F_\phi)}{\partial\theta} - \frac{\partial(rF_\theta)}{\partial\phi}\right]\mathbf{e}_r + \frac{1}{r\sin\theta}\left[\frac{\partial F_r}{\partial\phi} - \frac{\partial(r\sin\theta F_\phi)}{\partial r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\mathbf{e}_\phi\ .$$

Note that since r is independent of θ and ϕ, etc., we can for instance take the r outside the differentiations in the $\mathbf{e}_r$ component and cancel it with an r in the denominator. Remember the answer is a curl so it's a vector field. Do not add all the components together, forgetting the vectors $\mathbf{e}_r$ etc (this is a common error).

Note: the full expression above looks quite daunting. However in many problems this may simplify considerably using symmetry: for example, if a given problem is symmetrical around the z-axis, then we will have $F_\phi = 0$ and $\partial F_r/\partial\phi = 0$ and $\partial F_\theta/\partial\phi = 0$, so four of the six derivatives will vanish.

Exercise 5.3. Show by expanding it that the determinant definition is equivalent to the full expressions for the individual components given above. □

Exercise 5.4. What is $\nabla\times\mathbf{F}$ in cylindrical polar coordinates?

Note that if ρ and z have dimensions of length and ϕ is dimensionless (because it's an angle), then all the terms in the expression for $\nabla\times\mathbf{F}$ should have the same dimensions, namely the dimensions of $\mathbf{F}$ divided by length. This is a simple check that you should make. □

Exercise 5.5. Use spherical polar coordinates to evaluate the divergence and curl of $\mathbf{r}/r^3$. [Hint: don't forget that in spherical polar coordinates, the position vector $\mathbf{r}$ is equal to $r\mathbf{e}_r$.] □

Exercise 5.6. State Stokes's theorem, and verify it for the hemispherical surface $r = 1$, $z \geq 0$, with the vector field $\mathbf{A}(\mathbf{r}) = (y,-x,z)$. □

Exercise 5.7. The vector field $\mathbf{B}(\rho) = (0,\rho^{-1},0)$ in cylindrical polar coordinates (ρ,ϕ,z). Evaluate $\nabla\times\mathbf{B}$. Evaluate the line integral $\int_{\mathscr{C}}\mathbf{B}.\mathrm{d}\mathbf{r}$, where $\mathscr{C}$ is the unit circle $z = 0$, $\rho = 1$, $0 \leq \phi \leq 2\pi$. Does Stokes's theorem apply? □

Note: To conclude this chapter, we will note that many applied maths or Physics problems involve an expression like $\nabla^2 V$, where V is a scalar field and ∇^2 is the Laplacian operator, in cylindrical or spherical polar coordinates. We can get the expressions for $\nabla^2 V$ in polar coordinates using firstly the definition Eq. 3.10 (recall this was $\nabla^2 V \equiv \mathrm{div}(\mathrm{grad}\,V)$) , and then using Eq. 5.7 for grad V, then taking div of that with Eq. 5.8.

The results are available in most textbooks; you will not be expected to memorise those, but you might be given them in an exam question and asked to calculate something, so it's worth taking a look especially if you are taking applied maths courses later.

Appendix

Other ways of doing Example 5.5 are as follows

The second method is to divide the volume removed into two parts: (i) a cylinder with radius $\sin\theta_1$ and height $\cos\theta_1$, and (ii) a 'top-slice'. Volume (i), the cylinder, is easy: $2\pi\sin^2\theta_1\cos\theta_1$. To get volume (ii) we integrate over ρ and then z

$$4\pi\int_{\cos\theta_1}^{1} dz\int_0^{\sqrt{1-z^2}}\rho\, d\rho = 2\pi\int_{\cos\theta_1}^{1}(1-z^2)\, dz = \frac{2\pi}{3}(2+\cos^3\theta_1 - 3\cos\theta_1).$$

The sum of volumes (i) and (ii) is $\frac{4\pi}{3}(1-\cos^3\theta_1)$ as expected.

A third way also divides the volume removed into two parts: (i) an 'ice-cream cone' or cone with a spherical top, and (ii) a cylinder minus cone. The volume (i) is

$$4\pi\int_0^{\theta_1}\sin\theta\, d\theta\int_0^1 r^2\, dr = \frac{4\pi}{3}(1-\cos\theta_1).$$

Volume (ii), a cylinder with cone removed, is a bit harder:

$$4\pi\int_0^{\cos\theta_1} dz\int_{z\tan\theta_1}^{\sin\theta_1}\rho\, d\rho = 2\pi\int_0^{\cos\theta_1}(\sin^2\theta_1 - z^2\tan^2\theta_1)\, dz = \frac{4\pi}{3}\sin^2\theta_1\cos\theta_1$$

(which notice is $\frac{2}{3}$ of the volume of the cylinder). Again the sum of the volumes integrated is $\frac{4\pi}{3}(1-\cos^3\theta_1)$.

Finally, a fourth possibility is to integrate for the volume remaining after coring, which is

$$4\pi\int_0^{\cos\theta_1} dz\int_{\sin\theta_1}^{\sqrt{1-z^2}}\rho\, d\rho = 2\pi\int_0^{\cos\theta_1}(1-z^2-\sin^2\theta_1)\, dz = \frac{4\pi}{3}\cos^3\theta_1.$$

SUMMARY OF ORTHOGONAL CURVILINEAR COORDINATES

In orthogonal curvilinear coordinates (u_1, u_2, u_3), with corresponding unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3, the gradient of a scalar field V is given by

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\mathbf{e}_3 \qquad ;$$

the divergence of a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is given by

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_3 h_1 F_2) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right] \qquad ;$$

and the curl of the same vector field is given by

$$\nabla\times\mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix} \qquad .$$

Cartesian coordinates:
$(u_1, u_2, u_3) \equiv (x, y, z)$; arc-length parameters $h_1 = 1$, $h_2 = 1$, $h_3 = 1$.

Cylindrical polar coordinates:
$(u_1, u_2, u_3) \equiv (\rho, \phi, z)$; arc-length parameters $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

Spherical polar coordinates:
$(u_1, u_2, u_3) \equiv (r, \theta, \phi)$; arc-length parameters $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.