The Kronecker Delta and ϵ - δ Relationship

Techniques for more complicated vector identities

Overview

We have already learned how to use the Levi - Civita permutation tensor to describe cross products and to help prove vector identities. We will now learn about another mathematical formalism, the Kronecker delta, that will also aid us in computing vector products and identities.

Dot Product Redux

We have already seen in class how to write vectors in component notation and to take the dot product of those vectors. Let' s revisit this problem and start by noting that if we have two vectors **A** and **B,** they can be written in component form as :

$$\begin{aligned} \mathbf{A} &= A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \\ \mathbf{B} &= B_x \hat{x} + B_y \hat{y} + B_z \hat{z} \end{aligned} \tag{1}$$

Now, if we take the dot product of these vectors, we would write :

$$\mathbf{A} \cdot \mathbf{B} = \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z}\right) \cdot \left(B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\right) \tag{2}$$

If we did the complete term - by - term dot product multiplication of eq. (2), we would get nine terms :

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} = \; & A_x B_x \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} + A_x B_y \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} + A_x B_z \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} \\ & + A_y B_x \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} + A_y B_y \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} + A_y B_z \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} \\ & + A_z B_x \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} + A_z B_y \hat{\mathbf{z}} \cdot \hat{\mathbf{y}} + A_z B_z \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} \end{aligned} \tag{3}$$

We have learned in class that six of the terms in (3) will be non - zero, and that the only three non - zero terms are :

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

We can describe all the possible components of a dot product including both the zero and non - zero terms. For this, we would need a symbolic method of determining when the dot product of two unit vectors was 0 or 1. The simple way of showing this is with the **Kronecker delta**. The Kronecker delta, δ_{ij} is defined as:

$$\begin{aligned} \delta_{ij} &= 0 \text{ if } i \neq j \\ & \quad 1 \text{ if } i = j \text{ where i and j are subscripts} \end{aligned}$$

As you can see, the Kronecker delta nicely summarizes the rules for computing dot products of orthogonal unit vectors; if the two vectors have the same subscript, meaning they are in the same direction, their dot product is one. If they have different subscripts, meaning they are in different directions, their dot product is zero.

We can write an expression for the dot product between two vectors as :

$$\mathbf{A} \cdot \mathbf{B} = A_i B_j \delta_{ij} \tag{4}$$

You should be able to see that eq. (4) reduces quickly to the expression for dot product you are already familiar with. The expression in (4) is zero for unless i=j. Thus, we can set i=j in (4) and recover:

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i \tag{5}$$

as we learned from before. You should still write dot products in the simple form indicated in (5), and you should get so familiar with this that you recognize immediately that any product of the form $\Gamma_\alpha \aleph_\alpha$ represents the dot product between the two vectors Γ and $\aleph$ since there is a repeated index. The purpose of this exercise is to introduce you to the Kronecker delta notation.

The Kronecker Delta and Einstein Summation Notation

Recall that summation notation is built upon a simple protocol : repeated indices indicate a sum over that index from 1 to 3. Be sure to recognize that expressions like δ_{ij} do not imply any summation since there is no repeated index. Let's look at some examples of summation notation involving Kronecker deltas.

▪ Example 1 :

What is the value of δ_{ii}? This expression does have a repeated index, and means we should sum over all values of i from 1 to 3. This means that:

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3 \tag{6}$$

This is of course exactly the same result you would get from δ_{jj} or δ_{kk}. The choice of index is irrelevant, what matters is that the index is repeated.

▪ Example 2 :

What is the value of $\delta_{ij}\,\delta_{jk}$? We realize that the first delta will go to zero unless i=j; we can make that substitution in the second delta and contract the two deltas into one as:

$$\delta_{ij}\,\delta_{jk} = \delta_{ik} \tag{7}$$

The logic is straightforward: the first delta will be zero unless i = j, and the second delta will be zero unless j = k; this is equivalent to saying that the product is zero unless i = k, the result reflected on the right of (7)

▪ Example 3 :

How would we evaluate the expression $x_i\, x_j\, \delta_{ij}$? We now have two repeated indices, i and j, and we sum over both of them. Setting "j" as the "inner" variable and summing over that first (only indexing the "i' counter once "j" runs from 1 to 3):

$$\begin{aligned} x_i\, x_j\, \delta_{ij} = {} & x_1\, x_1\, \delta_{11} + x_1\, x_2\, \delta_{12} + x_1\, x_3\, \delta_{13} \\ & + x_2\, x_1\, \delta_{21} + x_2\, x_2\, \delta_{22} + x_2\, x_3\, \delta_{23} \end{aligned} \tag{8}$$

$$+ x_3 x_1 \delta_{31} + x_3 x_2 \delta_{32} + x_3 x_3 \delta_{33}$$
$$= x_1^{\,2} + x_2^{\,2} + x_3^{\,3}$$

Of course, there is no need (nor is it advisable) to do these summations explicitly; I show them here to help you with the transition from explicit summation notation to the implicit Einstein summation notation. We could have deduced the final result in (8) immediately by recognizing that the product of $x_i\, x_j\, \delta_{ij}$ will be zero unless i=j, and if i=j, our expression simplifies to $x_i\, x_j$.

The ϵ - δ Relationship

As we investigate more complicated vector products, especially those dealing with cross products of cross products (like **A×(B×C))**, we will make frequent use of the "ϵ-δ relationship." First of all, this is not the same "ϵ-δ" that you might have encountered in studying continuity issues in first semester calculus (you may be familiar with the phrase..."for every ϵ there is a δ"...)

The relationship we are studying here relates products of permutation tensors to a series of Kronecker deltas and bears no substantive connection with continuity studies; they just make use of the same $\Gamma\rho\epsilon\epsilon\kappa$ letters. The ϵ-δ relationship we use in vector analysis is:

$$\epsilon_{ijk}\, \epsilon_{imn} = \delta_{jm}\, \delta_{kn} - \delta_{jn}\, \delta_{km} \tag{9}$$

Let' s consider (9) carefully to recognize the conditions under which we can use this relationship. First, notice that the left is a product of permutation tensors (the ϵ symbols). Notice that there is one repeated index (in this case the "i" index) and that this index is in the same location in both tensors; in both tensors, the "i" index occupies the first index location. In order to make use of this ϵ - δ relationship, it is necessary that the repeated index appear in the same "slot" in both tensors. If there is a repeated index occupying different slots (e.g. $\epsilon_{ijk}\, \epsilon_{mni}$ where "i" is repeated but is in the first slot in the first tensor and the third slot in the second tensor), you will need to permute (reorder) the indices in one of the tensors so that the repeated index occupies the same slot in both. At the end of this section I will show an example of how to do that.

Looking at (9) again, notice how the order of indices on the left relate to the order of indices on the right. The first pair of deltas on the right $(\delta_{jm}\, \delta_{kn})$ have indices from the same slot position in their tensors, in other words δ_{jm} has both "second slot" indices, and δ_{kn} has both "third slot" indices. The latter pair of deltas, $\delta_{jn}\, \delta_{km}$ mix the indices, combining the second index from the first tensor ("j") with the third index of the second tensor ("n"); the second delta δ_{km} combines the third index from the first tensor ("k") with the second index of the second tensor ("m"). Some people find this language helpful in remembering the order of subscripts when using the ϵ-δ relationship:

"inner-inner outer-outer minus inner-outer outer-inner"

and some people don' t ... at any rate, (9) is a good equation to learn and use since we will make use of it frequently

What do we do if we encounter a product of permutation tensors where the repeated index is not in the same slot in both tensors? This is something we will encounter is many situations, so we need to establish our protocols for dealing with this. Let's consider the product of $\epsilon_{ijk}\, \epsilon_{mni}$. Since we have a repeated index, we know we can make use of the ϵ-δ relation, but we need to have "i" occupy the same slot in both tensors. We can do this by realizing that these indices are cyclic, that means we do not change the value of the tensor if we change the order of indices in a cylic permutation.

What does cyclic permutation mean? In its simplest meaning, a cyclic permutation is one in which you can reorder the elements without changing their relative locations. In this case, let's consider the indices of the second tensor "mni". If we repeat this cycle we get "m n i m n i". Any ordering of these elements that does not change the relative locations of the indices is cyclic, so in practical terms, the following permutations are cyclic:

$$\text{mni, nim, imn}$$

and do not change the value of the permuation tensor, in other words,

$$\epsilon_{mni} = \epsilon_{nim} = \epsilon_{imn} \tag{10}$$

However, the following permutations are anti - cyclic :

$$\text{inm, nmi, min}$$

so that :

$$\epsilon_{inm} = \epsilon_{nmi} = \epsilon_{min} = -\epsilon_{imn} \tag{11}$$

Remember that changing the order in the permutation tensor changes the sign of the value.

So, getting back to our original question, how do we handle a product like $\epsilon_{ijk}\,\epsilon_{mni}$? We simply permute the second index until the repeated index ("i" in this case) cycles into the first slot. This means we write:

$$\epsilon_{ijk}\,\epsilon_{mni} = \epsilon_{ijk}\,\epsilon_{nim} = \epsilon_{ijk}\,\epsilon_{imn}$$

Of course, you could have gone immediately to the final term, but I am showing you all the intermediate steps. Finally, our product becomes :

$$\epsilon_{ijk}\,\epsilon_{mni} = \epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km} \tag{12}$$

More Vector Proofs

Let' s use our new insights to prove some important vector identities.

■ Example 1

In class, and in the text, we saw that the triple product gives us the volume of a parallelepiped formed by three vectors, A, B, and C.We write this triple product as $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$. Griffiths, on page 7 notes that:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \tag{13}$$

Let' s see how to use vector identities to prove these equalities. The first step is to write **A·(B×C)** in summation notation. Notice that we are going to take the cross product of a vector, and then take the dot product of A with the vector produced by crossing B and C. We can write this in summation form as:

$$A_i\left(\epsilon_{ijk}\,B_j\,C_k\right) \tag{14}$$

Let' s look at (14) to make sure we understand what we have written. The term in parentheses is the expression for the i^{th} component of $\boldsymbol{B}\times\boldsymbol{C}$. The ϵ symbol should alert you to look for a cross product; the two scalars following the ϵ (here, B and C). In (14), we are computing the i^{th} component of the cross product since the two repeated indices are the j and k indices. (Review the last classnotes to be sure you see why this is the case.) So if the term in parentheses is the i^{th} component of the cross product, multiplying it by the i^{th} component of **A** does in fact represent the summation notation way of writing **A·(B×C)**.

Now we move on to proving the equalities in (13). Much of the power of using summation notation is that you convert vector expressions into scalar expressions; and scalars are easily manipulated since scalar multiplication is commutative. Using (14), we realize that the order of multiplication of scalars is irrelevant, so we can rewrite (14) as:

$$B_j \left(\epsilon_{jki}\, C_k\, A_i\right) \tag{15}$$

Why do we choose that order of operations (and also the order of indices for the permutation tensor)? This expressions suggests we are going to take the dot product of B with some new cross product. The cross product we have is expressed by

$$\epsilon_{jki}\, C_k\, A_i \tag{16}$$

The ϵ symbol alerts you that a cross product is being formed from the two vector components that follow; the order in which those components follow must match the order of indices in the ϵ term. Also, in order that we not change the sign of our product, we need to make sure that we are making a cyclic permutation, and those permutations are (ijk), (jki), and (kij). So, what we have in (15) is :

$$A_i \left(\epsilon_{ijk}\, B_j\, C_k\right) = B_j \left(\epsilon_{jki}\, C_k\, A_i\right) = B_j\, (C \times A)_j = B \cdot (C \times A) \tag{17}$$

We can make one more cylic permutation of the expression in (14) and write :

$$A_i \left(\epsilon_{ijk}\, B_j\, C_k\right) = C_k \left(\epsilon_{kij}\, A_i\, B_j\right) = C_k\, (A \times B)_k = C \cdot (A \times B) \tag{18}$$

and we have proven our identity.

▪ Example 2

We now derive the infamous "BAC-CAB" rule!

This will be our first foray into expressions with multiple cross products, so we will get to use our full arsenal of techniques. We want to prove the following important vector identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{B} \cdot \mathbf{A})\, \mathbf{C} - \mathbf{C}\, (\mathbf{A} \cdot \mathbf{B}) \tag{19}$$

Our first task is to convert the left hand side into summation notation. Since there are two cross products, we will need to do this in two steps. First, the cross product of **(B×C)** will produce a new vector, let's call it **D**, and this can be written as:

$$D = B \times C \;\Rightarrow\; D_i = \epsilon_{ijk}\, B_j\, C_k \tag{20}$$

This should be getting old hat; the i^{th} component of D is given by the expression in (20). Now, our final product, **A×D** produces another vector **G**, and we can write:

$$G = A \times D \;\Rightarrow\; G_m = \epsilon_{mni}\, A_n\, D_i \tag{21}$$

We have to be very careful about subscipts and indices here; To compute the components of our final vector, G, we need to take the appropriate cross product of A with D; this means that we need to write an expression for the components of G in terms of A and D. We use the index "i" for our D components, since we have just computed the "i" component of D, and it is these components that will be crossed with A. However, we must use new indicies for G and A since we are doing a separate summation when we cross D with A.

Notice again the pattern of subscripts; the m^{th} component of G is produced from the expression $\epsilon_{mni}\, A_n\, D_i$. The ϵ term indicates a cross product of the two scalars that follow (A and D); the order of the subscripts in ϵ matches the order of subscripts of the components (A and D). The remaining subscript in the ϵ term is the component of the vector that is produced, so it is correct that the $\epsilon_{mni} A_n D_i$ term produces the m^{th} component of G.

Knowing that D_i can be written as shown in (20), we substitute this expression for D_i into (21) and get:

$$G_m = \epsilon_{mni} A_n D_i = \epsilon_{mni} A_n \left(\epsilon_{ijk} B_j C_k\right) \tag{22}$$

Since all the terms in (22) are scalars, we can reorder them as we wish and write :

$$G_m = \epsilon_{mni} A_n \left(\epsilon_{ijk} B_j C_k\right) = \epsilon_{mni} \epsilon_{ijk} A_n B_j C_k \tag{23}$$

And we have two ϵ terms on the right. We permute the first ϵ term so that the "i" index is in the first slot and get :

$$G_m = \epsilon_{imn} \epsilon_{ijk} A_n B_j C_k \tag{24}$$

and now we use the ϵ - δ rule :

$$G_m = \left(\delta_{jm} \delta_{kn} - \delta_{mk} \delta_{nj}\right) A_n B_j C_k = \delta_{jm} \delta_{kn} A_n B_j C_k - \delta_{mk} \delta_{nj} A_n B_j C_k \tag{25}$$

Remember, the terms in these expressions are scalars and can be multiplied in any order.

Now, let' s consider the terms on the right in (25). In the first set of terms, $\delta_{jm} \delta_{kn} A_n B_j C_k$, we realize that the expression will be zero unless j=m and also k=n. This means that the only nonzero terms in this expression are $A_n B_m C_n$.

For the second set of terms, $\delta_{mk} \delta_{nj} A_n B_j C_k$, the nonzero terms arise only if k=m and also j=n. This means that the only nonzero terms in this expression are $A_n B_n C_m$. Combining these results and substituting back into (25) we have:

$$G_m = A_n B_m C_n - A_n B_n C_m \tag{26}$$

Noting again that the terms in (26) can be multiplied in any order, we rearrange this to the form :

$$G_m = B_m A_n C_n - C_m A_n B_n \tag{27}$$

And now we are almost done. The product $A_n C_n$ is simply the dot product A·C. Similarly, $A_n B_n$ is the dot product A·B. This allows us to write:

$$G_m = B_m (A \cdot C) - C_m (A \cdot B) \tag{28}$$

so when we sum over all coordinates, this becomes :

$$G = G_m \hat{e}_m = B_m (A \cdot C) \hat{e}_m - C_m (A \cdot B) \hat{e}_m \tag{29}$$

and FINALLY,

$$G = B (A \cdot C) - C (A \cdot B) \tag{30}$$

Q E D !!