MTH5102 Calculus III: Revision for the exam

Lecturer: Prof. M.A.H. MacCallum

December 2008

New format and its implications

The format will change this year in accordance with the School's new examinations policy. In the new format, (i) students will have to answer **all** questions for full marks and (ii) there should be enough material of a straightforward character to give a pass mark.

For Calculus III, I will meet the second requirement by following the previous practice of having 60% relatively straightforward material (at the level of the old Section A), but some of this may be in the form of the introductory parts of longer questions (whose total length may be up to the length of the old Section B questions).

Since the syllabus should be covered (though some of the points may be considered to have been adequately covered by the coursework and mid-term test), the first requirement implies that it will not be possible to get a high mark without studying the later parts of the course. Whereas, for example, in 2008 a student who had studied only chapters 1-5 could get a maximum mark of 83 – by perfect answers to questions A1-5 and B1-2 – the same student would only be able to get something between 60 and 70 this year, since chapters 6-8 are at least 30% of the total material.

A sample paper based on questions from the 2007 and 2008 exams has been provided. Note that with only 40% of the total mark in the past coming from longer questions, and quite a number of topics which can lead to longer questions, the choice of longer questions is unlikely to be the same every year (and hence not necessarily the same as on the sample paper).

Possible topics for longer questions would be those covered by section B questions on past exams. Questions such as B3 on the 2007 or 2008 papers, which combine material from different parts of the course in a way not directly covered by lectures, are less likely to occur.

The new format limits the options, which may help guide your revision. For instance, most Frobenius series questions one could set would be worth 15 marks or more, so the question(s) on chapter 6 would probably be either a (shorter) question on Picard series and/or a simple Sturm-Liouville question, or a (longer) Frobenius series, not both a Picard and a Frobenius problem. There are few ways of setting a short question on Laplace's equation which covers the main ideas.

Content

Here I note the main things to be revised in each chapter, followed by some comments on possible questions. The sheet already issued, sorting past questions by topic, will guide you to suitable problems to use for practice. (Ignore the ones marked 'Old syllabus': these are mentioned only so that when looking at an old paper you do not start to worry that you cannot do them!)

Note that the gaps in questions on some of the topics for some years arise from changes of syllabus: you should then be able to find further questions in related courses. For example there are sample questions on Frobenius and Picard methods in Calculus II up to 2006 (the topic had been moved from Calculus III to Calculus II in 2002).

Chapter 1:
Equations of curves: parametrization of simple cases.
Equations of standard surfaces: sphere, ellipsoid, paraboloid, hyperboloid.
Equations of lines and planes.
Gradient of a scalar: definition and evaluation. Geometric meaning and use.

This preliminary material is unlikely to be suitable for a long question. Despite the overlap with Calculus II, it is common to set a question like A1 in 2007: this asked you to work out a gradient, identify a surface and find a tangent plane.

The following topics will not be tested as questions in themselves, but may arise in the course of questions on later material: trigonometric and hyperbolic functions, vector algebra including dot and cross products, evaluation of double and triple integrals.

Chapter 2:
Differentiation of a vector with respect to a parameter. (Could be tested only as part of a problem on line integrals.)
Divergence and curl: definition and evaluation. Geometric interpretation. (Curl could be tested as part of a question on conservative fields.)

If you need any formula for derivatives of products which is not obviously of the form you would expect from the Leibniz rule and rules of vector products, it will be given (there are two such rules, marked in the lecture notes). You should also remember $\nabla \times \nabla \Phi = \mathbf{0}$ and $\nabla \cdot \nabla \times \mathbf{F} = 0$.

Vector derivatives could appear in both short and long questions, and could be combined with index notation and/or curvilinear coordinates.

Chapter 3:
Definition of line integral. Evaluation of simple cases.
Definition of surface integral. Calculation of the vector d**S**. Evaluation of such integrals, directly or using integral theorems.
Evaluation of volume integrals.
Statements of Divergence, Green's and Stokes's Theorems. (Pay attention to the required directions of normals and curves.) Examples applying these theorems.
Definition of a conservative $\mathbf{F} \Leftrightarrow \nabla \times \mathbf{F} = \mathbf{0} \Leftrightarrow \mathbf{F} = \nabla\Phi$ for some Φ. Finding Φ for given $\mathbf{F}$.

Direct evaluation of surface integrals (as distinct from evaluations using integral theorems) is generally too long for a short question.

Chapter 4:
Rules for the summation convention.
Definition of Kronecker delta: index substitution property.
Definition of Levi-Civita epsilon: properties under permutation of indices.
Applications of these rules.

If the identity for $\varepsilon_{ijk}\varepsilon_{ilm}$ is needed, it will be given.

Chapter 5:
Definitions of cylindrical and spherical polars.
How to check orthogonality.
Applying to specific cases the formulae given for grad, div and curl in general orthogonal coordinates.

This could be combined with use of the Divergence theorem, derivation of the form of Laplace's equation in polar coordinates, and so on. Note that the formulae for the general case will appear on the front page of the question paper: make sure you understand them!

Chapter 6:
Picard and Frobenius methods.
Derivation of Legendre polynomials.
Sturm-Liouville systems: orthogonality theorem.

See the remarks above about this chapter. If you are set a Frobenius problem, you will be given formulae (6.14) and (6.15) of the notes if it would help you to shorten the calculation (see the coursework examples of how these can be applied).

Chapter 7:

Full-range Fourier series: you need to understand the ideas and remember the formulae for the coefficients.
Half-range series: odd and even functions. Know the formulae for the coefficients and note that effort can be saved in many examples for full range series by checking whether the functions are odd or even or can be made so.
Convergence properties: value of the series where the function has a discontinuity. Parseval's theorem (if you need this, the formula will be given).
Arbitrary range series, using change of variable.

The functions you might be asked to work with will be (possibly sums of) the functions 1, x and x^2 (where the integrals may need integration by parts), trigonometric functions (where you need trigonometric identities), or exponential or hyperbolic functions. For the last case, where you need identities like

$$\int_0^\pi e^x \cos(nx)\mathrm{d}x = \left[\frac{e^x \cos(nx)}{n^2}\right]_0^\pi - \frac{1}{n^2}\int_0^\pi e^x \cos(nx)\mathrm{d}x,$$

any such identity required would be given.

Chapter 8:

Laplace's equation: uniqueness of solutions.
Separation of variables for Cartesian coordinates, and z-independent cylindrical and axisymmetric spherical harmonics.

Only Dirichlet boundary conditions will be used in examples.
Problems may involve use of a series of separable solutions to solve rectangular boundary value problems in Cartesian coordinates, and include use of Fourier series for the boundary data.
One could also set problems requiring use of a series of separable solutions to solve (a) z-independent cylindrical and (b) axisymmetric spherical problems.
Uniqueness is likely to be required only in justifiying the answers.

If you are set a question whose answer requires a Fourier series (e.g. on one side of a rectangular boundary) you will be given the Fourier series or one related to it by rescaling a variable.