Change of basis, reciprocal basis vectors, covariant and contravariant components of a vector and metric tensor

Math 1550 lecture notes, Prof. Anna Vainchtein

1 Change of basis

Recall that any n linearly independent vectors in $\mathbb{R}^n$ form a basis in $\mathbb{R}^n$. In what follows, we will mostly consider $n = 3$ and sometimes $n = 2$.

Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ and $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ be two sets of bases in $\mathbb{R}^3$. For example, we can consider

$$\mathbf{e_1} = \mathbf{i_1} + 2\mathbf{i_2}, \quad \mathbf{e_2} = \mathbf{i_1} + 2\mathbf{i_2} + \mathbf{i_3}, \quad \mathbf{e_3} = -\mathbf{i_2} + \mathbf{i_3} \tag{1}$$

and

$$\mathbf{u_1} = \mathbf{i_1} + \mathbf{i_2}, \quad \mathbf{u_2} = \mathbf{i_2} + 2\mathbf{i_3}, \quad \mathbf{u_3} = 2\mathbf{i_1} + \mathbf{i_2} - \mathbf{i_3}, \tag{2}$$

where $\{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}\}$ is the standard basis in $\mathbb{R}^3$. These two sets forms a basis in $\mathbb{R}^3$ because the vectors in each set are linearly independent. Indeed,

$$(\mathbf{e_1}\times\mathbf{e_2})\cdot\mathbf{e_3} = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 1 \neq 0, \qquad (\mathbf{u_1}\times\mathbf{u_2})\cdot\mathbf{u_3} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 1 \neq 0.$$

Suppose we want to change from the *old basis* $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ to the *new basis* $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$. We can express each of the new basis vectors in terms the old ones as follows:

$$\mathbf{u_i} = \alpha_i^1\mathbf{e_1} + \alpha_i^2\mathbf{e_2} + \alpha_i^3\mathbf{e_3}, \quad i = 1, 2, 3, \tag{3}$$

where

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \alpha_1^3 \\ \alpha_2^1 & \alpha_2^2 & \alpha_2^3 \\ \alpha_3^1 & \alpha_3^2 & \alpha_3^3 \end{bmatrix}$$

is the *matrix of the coefficients of the direct transformation* from old to new basis. Its ith row is the coordinates of $\mathbf{u_i}$ in the old basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. To find $\boldsymbol{\alpha}$ for given $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ and $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$, we need to solve the linear system (3). For example, with the bases given by (1) and (2), the relation (3) becomes

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \alpha_1^1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_1^2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \alpha_1^3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad i = 1, \tag{4}$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \alpha_2^1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2^2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \alpha_2^3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad i = 2, \tag{5}$$

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \alpha_3^1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_3^2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \alpha_3^3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad i = 3. \tag{6}$$

Rather than solving each the above systems of linear equations separately, we can combine (4)-(6) into a single system

$$\mathbf{U} = \boldsymbol{\alpha}\mathbf{E}, \tag{7}$$

where $\mathbf{U}$ and $\mathbf{E}$ are the matrices whose rows are $\mathbf{u_i}$ and $\mathbf{e_i}$, respectively. In our example we obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \alpha_1^3 \\ \alpha_2^1 & \alpha_2^2 & \alpha_2^3 \\ \alpha_3^1 & \alpha_3^2 & \alpha_3^3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solving this linear system to find

$$\boldsymbol{\alpha} = \mathbf{U}\mathbf{E}^{-1} \tag{8}$$

(why is $\mathbf{E}$ invertible for any basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$?), we obtain (do this as an exercise)

$$\boldsymbol{\alpha} = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 3 & -1 \\ 6 & -4 & 3 \end{bmatrix}. \tag{9}$$

For instance (read off the coefficients in the first row of $\boldsymbol{\alpha}$), we have $\alpha_1^1 = 2$, $\alpha_1^2 = -1$, $\alpha_1^3 = 1$, so that $\mathbf{u_1} = 2\mathbf{e_1} - \mathbf{e_2} + \mathbf{e_3}$ (verify that these indeed satisfy (4)).

Now, just as we expressed the new basis $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ in terms of the old basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, we can do the reverse via the *inverse transformation*

$$\mathbf{E} = \boldsymbol{\alpha}^{\mathbf{-1}}\mathbf{U}, \tag{10}$$

or

$$\mathbf{e_i} = (\alpha^{-1})_i^1\mathbf{u_1} + (\alpha^{-1})_i^2\mathbf{u_2} + (\alpha^{-1})_i^3\mathbf{u_3}, \quad i = 1, 2, 3, \tag{11}$$

where

$$\boldsymbol{\alpha}^{\mathbf{-1}} = \mathbf{E}\mathbf{U}^{-\mathbf{1}} \tag{12}$$

is the inverse of $\boldsymbol{\alpha}$ in (8). In our example it is given by (show)

$$\boldsymbol{\alpha}^{\mathbf{-1}} = \begin{bmatrix} 5 & -1 & -2 \\ 3 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}. \tag{13}$$

For example, $(\alpha^{-1})_2^1 = 3$, $(\alpha^{-1})_2^2 = 0$ and $(\alpha^{-1})_2^3 = -1$ (second row of $\boldsymbol{\alpha}^{\mathbf{-1}}$), so that at $i = 2$ (11) yields $\mathbf{e_2} = 3\mathbf{u_1} - \mathbf{u_3}$.

2 Reciprocal bases, covariant and contravariant components

Recall that a basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is called *orthogonal* if the basis vectors are perpendicular to one another:

$$\mathbf{e_i} \cdot \mathbf{e_j} = 0, \quad i \neq j.$$

If in addition each basis vector is a unit vector, $|\mathbf{e_i}| = 1$, the basis is called *orthonormal*. Thus, an orthonormal basis satisfies

$$\mathbf{e_i} \cdot \mathbf{e_j} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

The standard basis $\{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}\}$ is an orthonormal basis, but, for example, the basis $\{\mathbf{e_1} = \mathbf{i_1} + 2\mathbf{i_2}, \mathbf{e_2} = 2\mathbf{i_1} - \mathbf{i_2}, \mathbf{e_3} = 7\mathbf{i_3}\}$ is merely orthogonal (verify).

In any *orthonormal* basis, the components of any given vector $\mathbf{a}$ are the projections of the vector on the coordinate axes. Recalling that the projection of $\mathbf{a}$ on the kth axis along the unit vector $\mathbf{i_k}$ is the scalar product $\mathbf{a} \cdot \mathbf{i_k}$, we obtain

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{i_1})\mathbf{i_1} + (\mathbf{a} \cdot \mathbf{i_2})\mathbf{i_2} + (\mathbf{a} \cdot \mathbf{i_3})\mathbf{i_3}. \tag{14}$$

Importantly, this **does not hold for a non-orthonormal basis**, i.e.

$$\mathbf{a} \neq (\mathbf{a} \cdot \mathbf{e_1})\mathbf{e_1} + (\mathbf{a} \cdot \mathbf{e_2})\mathbf{e_2} + (\mathbf{a} \cdot \mathbf{e_3})\mathbf{e_3}.$$

unless the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ happens to be orthonormal. Indeed, for an *orthogonal* basis we have to first make the vectors unit by dividing each by its length, obtaining instead of (14)

$$\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{e_1}}{|\mathbf{e_1}|^2}\mathbf{e_1} + \frac{\mathbf{a} \cdot \mathbf{e_2}}{|\mathbf{e_2}|^2}\mathbf{e_2} + \frac{\mathbf{a} \cdot \mathbf{e_3}}{|\mathbf{e_3}|^2}\mathbf{e_3}.$$

Now observe that we *could express this in the same form* as (14) if we define a *new set of vectors*

$$\mathbf{e^1} = \frac{\mathbf{e_1}}{|\mathbf{e_1}|^2}, \quad \mathbf{e^2} = \frac{\mathbf{e_2}}{|\mathbf{e_2}|^2}, \quad \mathbf{e^3} = \frac{\mathbf{e_3}}{|\mathbf{e_3}|^2} \tag{15}$$

and write

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{e^1})\mathbf{e_1} + (\mathbf{a} \cdot \mathbf{e^2})\mathbf{e_2} + (\mathbf{a} \cdot \mathbf{e^3})\mathbf{e_3}$$

But this means that $a^i = \mathbf{a} \cdot \mathbf{e^i}$ are the components of the vector $\mathbf{a}$ in the orthogonal basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, i.e.

$$\mathbf{a} = a^1\mathbf{e_1} + a^2\mathbf{e_2} + a^3\mathbf{e_3}, \quad a^i = \mathbf{a} \cdot \mathbf{e^i} \tag{16}$$

These components are called *contravariant* components of $\mathbf{a}$.

Suppose now the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is not even orthogonal. Can we generalize the definition (15) of the new vectors, so that (16) still works? The answer is yes. To do this, we need to introduce *reciprocal vectors* $\{\mathbf{e^1}, \mathbf{e^2}, \mathbf{e^3}\}$, which are related to the original basis in the following way:

$$\mathbf{e_i} \cdot \mathbf{e^j} = \delta_i^j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{17}$$

So the jth reciprocal basis vector $\mathbf{e^j}$ is orthogonal to the vectors $\mathbf{e_i}$, $i \neq j$, and forms an acute angle with $\mathbf{e_j}$ (why?). Note that the vectors (15) we introduced for an orthogonal basis satisfy these relations. However, (15) only defines reciprocal vectors for the special case of an orthogonal basis.

We will derive more general formulas below. Now, observe that, using (16) and (17), we obtain

$$\begin{aligned}\mathbf{a}\cdot\mathbf{e^1} = (a^1\mathbf{e_1} + a^2\mathbf{e_2} + a^3\mathbf{e_3})\cdot\mathbf{e^1} &= a^1\mathbf{e_1}\cdot\mathbf{e^1} + a^2\mathbf{e_2}\cdot\mathbf{e^1} + a^3\mathbf{e_3}\cdot\mathbf{e^1}\\ &= a^1\cdot 0 + a^2\cdot 0 + a^3\cdot 1 = a^1.\end{aligned}$$

Similarly, one can see that $\mathbf{a}\cdot\mathbf{e^2} = a^2$ and $\mathbf{a}\cdot\mathbf{e^3} = a^3$. Thus $a^i = \mathbf{a}\cdot\mathbf{e^i}$ and (16) holds.

It remains to determine the reciprocal basis for a given (not necessarily orthogonal) basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. Consider $\mathbf{e^1}$. First of all, by (17), it must be orthogonal to both $\mathbf{e_2}$ and $\mathbf{e_3}$, so it must be a multiple of $\mathbf{e_2}\times\mathbf{e_3}$:

$$\mathbf{e^1} = \beta(\mathbf{e_2}\times\mathbf{e_3}),$$

where β is a number to be determined. To find it, recall that we must also have $\mathbf{e^1}\cdot\mathbf{e_1} = 1$, so $\beta\mathbf{e_1}\cdot(\mathbf{e_2}\times\mathbf{e_3}) = 1$, yielding $\beta = 1/V$, where

$$V = \mathbf{e_1}\cdot(\mathbf{e_2}\times\mathbf{e_3}) \tag{18}$$

is the signed volume of the parallelipiped spanned by the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. We have $V > 0$ if the basis is right-handed and $V < 0$ otherwise. So we have

$$\mathbf{e^1} = \frac{\mathbf{e_2}\times\mathbf{e_3}}{V}.$$

The expressions for $\mathbf{e^2}$ and $\mathbf{e^3}$ are derived similarly. In general, we have

$$\mathbf{e^i} = \frac{\mathbf{e_j}\times\mathbf{e_k}}{V}, \tag{19}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, i.e. if $i = 1$, $(j, k) = (2, 3)$, if $i = 2$, $(j, k) = (3, 1)$ and if $i = 3$, $(j, k) = (1, 2)$ in (19). Note that

$$V = \mathbf{e_i}\cdot(\mathbf{e_j}\times\mathbf{e_k})$$

due to the properties of the scalar triple product. Now, if the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is *orthonormal*, the reciprocal vectors *coincide* with the original basis vectors, $\mathbf{e^i} = \mathbf{e_i}$, because in this case $V = \pm 1$ and $\mathbf{e_i} = \mathrm{sign}(V)\mathbf{e_j}\times\mathbf{e_k}$ ($V = 1$ for a right-handed basis). If the basis is *orthogonal* (but not necessarily orthonormal), then

$$V = \pm|\mathbf{e_i}||\mathbf{e_j}||\mathbf{e_k}|, \quad \mathbf{e_j}\times\mathbf{e_k} = \mathrm{sign}(V)|\mathbf{e_j}||\mathbf{e_k}|\frac{\mathbf{e_i}}{|\mathbf{e_i}|}$$

(why?), so that (19) reduces in this case (and in this case only!) to (15).

As an example, consider the basis

$$\mathbf{e_1} = 2\mathbf{i_1} + \mathbf{i_2}, \quad \mathbf{e_2} = \mathbf{i_1} + 2\mathbf{i_2}, \quad \mathbf{e_3} = \mathbf{i_3}. \tag{20}$$

Note that it is *not* orthogonal because $\mathbf{e_1} \cdot \mathbf{e_2} = 4 \neq 0$. In this case we have

$$V = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3,$$

and obtain

$$\mathbf{e^1} = \frac{1}{3} \begin{vmatrix} \mathbf{i_1} & \mathbf{i_2} & \mathbf{i_3} \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{2}{3}\mathbf{i_1} - \frac{1}{3}\mathbf{i_2},$$

$$\mathbf{e^2} = \frac{1}{3} \begin{vmatrix} \mathbf{i_1} & \mathbf{i_2} & \mathbf{i_3} \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -\frac{1}{3}\mathbf{i_1} + \frac{2}{3}\mathbf{i_2}$$

and $\mathbf{e^3} = \mathbf{e_3} = \mathbf{i_3}$ (why?). See Fig. 1, where $\mathbf{e_1}$ and $\mathbf{e_2}$ are shown in green, and the reciprocal vectors $\mathbf{e^1}$ and $\mathbf{e^2}$ are shown in blue. Note that by construction $\mathbf{e^1}$ is orthogonal to $\mathbf{e_2}$, and $\mathbf{e^2}$ is orthogonal to $\mathbf{e_1}$.

For a given vector $\mathbf{a}$, say,

$$\mathbf{a} = 2\mathbf{i_1} + 3\mathbf{i_2},$$

we can now use (16) to find its contravariant components in the basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. We get

$$a^1 = \mathbf{a} \cdot \mathbf{e^1} = \frac{1}{3}, \quad a^2 = \mathbf{a} \cdot \mathbf{e^2} = \frac{4}{3}, \quad a^3 = \mathbf{a} \cdot \mathbf{e^3} = 0,$$

so that $\mathbf{a} = \frac{1}{3}\mathbf{e_1} + \frac{4}{3}\mathbf{e_2}$. See Fig. 1.

Now observe that the original basis vectors are reciprocal of the reciprocal ones. Thus we can just as well expand the same vector $\mathbf{a}$ along the reciprocal basis vectors:

$$\mathbf{a} = a_1\mathbf{e^1} + a_2\mathbf{e^2} + a_3\mathbf{e^3}, \quad a_i = \mathbf{a} \cdot \mathbf{e_i}. \tag{21}$$

The components a_i are called the *covariant* components of $\mathbf{a}$. In our example, we obtain

$$a_1 = \mathbf{a} \cdot \mathbf{e_1} = 7, \quad a_2 = \mathbf{a} \cdot \mathbf{e_2} = 8, \quad a_3 = \mathbf{a} \cdot \mathbf{e_3} = 0,$$

so we have $\mathbf{a} = 7\mathbf{e^1} + 8\mathbf{e^2}$. See Fig. 2.

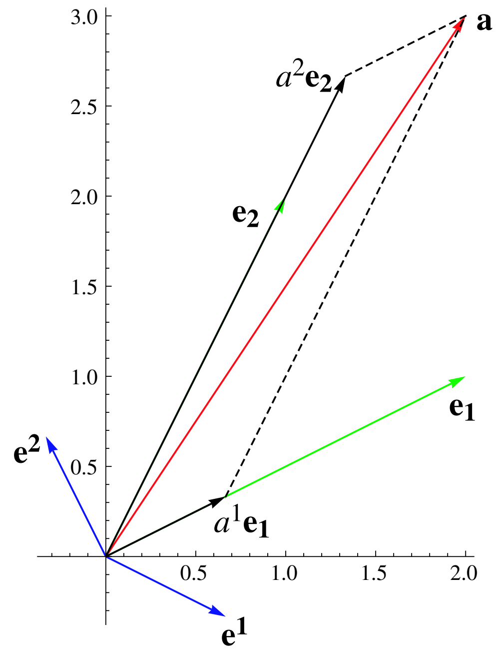

Figure 1: Basis vectors, reciprocal basis vectors and contravariant components

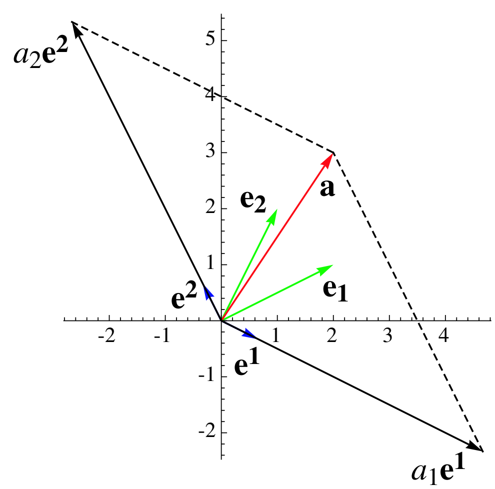

Figure 2: Basis vectors, reciprocal basis vectors and covariant components

Change of basis, revisited. Consider now a new basis, $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and recall that the new basis vectors $\mathbf{u_i}$ can be expressed in terms of the old ones via (3). But now we recognize that the coefficients α_i^k entering (3) are just the contravariant components of $\mathbf{u_i}$ in the basis of $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$! Thus we have

$$\alpha_i^k = \mathbf{u_i} \cdot \mathbf{e^k}. \tag{22}$$

Similarly, (11) implies that

$$(\alpha^{-1})_i^k = \mathbf{e_i} \cdot \mathbf{u^k}. \tag{23}$$

Exercise 1. Show that the reciprocal basis for $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ defined in (1) is given by

$$\mathbf{e^1} = 3\mathbf{i_1} - \mathbf{i_2} - \mathbf{i_3}, \quad \mathbf{e^2} = -2\mathbf{i_1} + \mathbf{i_2} + \mathbf{i_3}, \quad \mathbf{e^3} = 2\mathbf{i_1} - \mathbf{i_2}$$

and the reciprocal basis of $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ defined in (2) is

$$\mathbf{u^1} = -3\mathbf{i_1} + 4\mathbf{i_2} - 2\mathbf{i_3}, \quad \mathbf{u^2} = \mathbf{i_1} - \mathbf{i_2} + \mathbf{i_3}, \quad \mathbf{u^3} = 2\mathbf{i_1} - 2\mathbf{i_2} + \mathbf{i_3}.$$

Use these results, (1), (2), (22) and (23) to verify (9) and (13).

How do the reciprocal vectors transform under the change of basis? To see this, express $\mathbf{u^i}$ in terms of the reciprocal basis $\{\mathbf{e^1}, \mathbf{e^2}, \mathbf{e^3}\}$. Since we are using the reciprocal basis, the components are going to be *covariant*, i.e. we need to take dot products with $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$:

$$\mathbf{u^i} = (\mathbf{u^i} \cdot \mathbf{e_1})\mathbf{e^1} + (\mathbf{u^i} \cdot \mathbf{e_2})\mathbf{e^2} + (\mathbf{u^i} \cdot \mathbf{e_3})\mathbf{e^3}, \quad i = 1, 2, 3.$$

But from (23) we can see that $\mathbf{u^i} \cdot \mathbf{e_k} = (\alpha^{-1})_k^i$, and thus the ith reciprocal basis vector of the new basis is related to the reciprocal vectors of the old basis via the coefficients in the ith *column* of the *inverse* transformation (12):

$$\mathbf{u^i} = (\alpha^{-1})_1^i \mathbf{e^1} + (\alpha^{-1})_2^i \mathbf{e^2} + (\alpha^{-1})_3^i \mathbf{e^3}, \quad i = 1, 2, 3. \tag{24}$$

Similarly, going back from the new reciprocal basis to the old one involves the ith column of the *direct* transformation (8):

$$\mathbf{e^i} = \alpha_1^i \mathbf{u^1} + \alpha_2^i \mathbf{u^2} + \alpha_3^i \mathbf{u^3}, \quad i = 1, 2, 3. \tag{25}$$

Compare this to (3) and (11) that involve the ith *row* of direct and inverse transformations, respectively.

In fact, define matrices $\mathbf{U^r}$ and $\mathbf{E^r}$ whose *columns* are the reciprocal basis vectors $\{\mathbf{u^1}, \mathbf{u^2}, \mathbf{u^3}\}$ and $\{\mathbf{e^1}, \mathbf{e^2}, \mathbf{e^3}\}$, respectively (recall that to form $\mathbf{U}$ and $\mathbf{E}$ we put the original basis vectors as *rows*). Then we have

$$\mathbf{U}\mathbf{U^r} = \mathbf{E}\mathbf{E^r} = \mathbf{I},$$

since, by the definition (17), the scalar product of the ith row of $\mathbf{E}$ ($\mathbf{U}$) and the jth column of $\mathbf{E^r}$ ($\mathbf{U^r}$) is 1 if $i = j$ and zero otherwise. The relations (24) and (25) can then be written as

$$\mathbf{U^r} = \mathbf{E^r}\boldsymbol{\alpha}^{-1}, \quad \mathbf{E^r} = \mathbf{U^r}\boldsymbol{\alpha}.$$

In our example from Exercise 1, the first of these gives

$$\begin{bmatrix} -3 & 1 & 2 \\ 4 & -1 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 \\ -1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -1 & -2 \\ 3 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}.$$

Exercise 2. Verify the above identity.

The relation between old and new bases and the reciprocal bases is summarized in the diagram in Fig. 3.

3 Why covariant and contravariant?

In the previous section, we introduced covariant and contravariant components of a vector $\mathbf{a}$ via (21) and (16). Note that unless the basis is *orthonormal*, the two sets of components are *different*. To see why they are called covariant and contravariant, we must consider how they transform under a change of basis. For covariant components of $\mathbf{a}$ in the new basis, we have

$$\mathbf{a} = \tilde{a}_1\mathbf{u^1} + \tilde{a}_2\mathbf{u^2} + \tilde{a}_3\mathbf{u^3},$$

with

$$\tilde{a}_i = \mathbf{a} \cdot \mathbf{u_i} = (a_1\mathbf{e^1} + a_2\mathbf{e^2} + a_3\mathbf{e^3}) \cdot \mathbf{u_i} = \alpha_i^1 a_1 + \alpha_i^2 a_2 + \alpha_i^3 a_3,$$

where we used (21) and (22). Comparing this to (3), we see that the covariant components transform via the *direct* transformation, the same way the

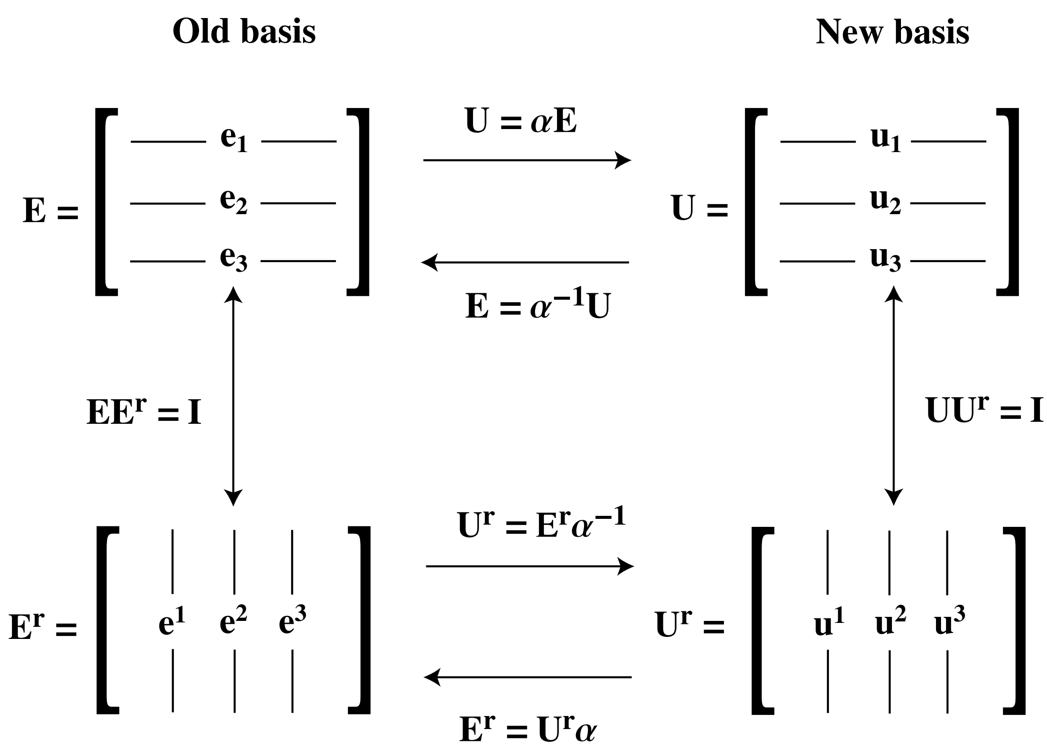

Figure 3: Basis vectors, reciprocal basis vectors and change of basis

old basis transforms into the new. This is why they are called **co**variant. Meanwhile, for contravariant components of **a** in the new basis we have

$$\mathbf{a} = \tilde{a}^1\mathbf{u_1} + \tilde{a}^2\mathbf{u_2} + \tilde{a}^3\mathbf{u_3},$$

with

$$\tilde{a}^i = \mathbf{a}\cdot\mathbf{u^i} = (a^1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3})\cdot\mathbf{u^i} = (\alpha^{-1})^i_1 a^1 + (\alpha^{-1})^i_2 a^2 + (\alpha^{-1})^i_3 a^3,$$

where we used (16) and (23). Compare this to (24): the contravariant components transform as the reciprocal bases, via the *inverse* transformation, which is the *opposite* of how the original old basis transforms into the new one. Thus they are called **contra**variant.

4 Relation between covariant and contravariant components: metric tensor

Observe that

$$a_i = \mathbf{a}\cdot\mathbf{e_i} = (a^1\mathbf{e_1} + a^2\mathbf{e_2} + a^3\mathbf{e_3})\cdot\mathbf{e_i} = \sum_{k=1}^{3} g_{ik}a^k$$

and

$$a^i = \mathbf{a}\cdot\mathbf{e^i} = (a_1\mathbf{e^1} + a_2\mathbf{e^2} + a_3\mathbf{e^3})\cdot\mathbf{e^i} = \sum_{k=1}^{3} g^{ik}a_k,$$

where

$$g_{ik} = \mathbf{e_i}\cdot\mathbf{e_k}, \quad g^{ik} = \mathbf{e^i}\cdot\mathbf{e^k} \tag{26}$$

are covariant and contravariant components of *metric tensor* (we will discuss what that means later). This tensor is symmetric since one can see that $g_{ik} = g_{ki}$ and $g^{ik} = g^{ki}$. As we showed in class, the metric defined by a given set of coordinates (x^1, x^2, x^3), with position vector of a point given by $\mathbf{r} = x^1\mathbf{e_1} + x^2\mathbf{e^2} + x^3\mathbf{e_3}$ is

$$(ds)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}dx^i dx^j \tag{27}$$

In the case of Cartesian coordinates with an orthonormal basis we have $g_{ik} = \delta_{ik}$, so that the metric reduces to

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

This, however, is not true for general oblique or curvilinear coordinates. Note also that if the basis is *orthogonal*, the off-diagonal covariant components g_{ik}, $i \neq k$, of the metric tensor in (26) are all zero, while the diagonal components are $g_{ii} = |\mathbf{e_i}|^2$ (no sum), so that (27) simplifies to

$$(ds)^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2,$$

where $h_i = \sqrt{g_{ii}} = |\mathbf{e_i}|$, $i = 1, 2, 3$ (no sum) are the metric coefficients. For instance, in cylindrical coordinates we have $x^1 = r$, $x^2 = \phi$, $x^3 = z$ and $h_1 = h_3 = 1$, $h_2 = r$ (will derive these later), so that $(ds)^2 = (dr)^2 + (rd\phi)^2 + (dz)^2$.

For a given basis, we can calculate both covariant and contravariant components of a metric tensor, g_{ik} and g^{ik}. In fact, as we have shown in class, they are related by

$$g^{ik} = \frac{G^{ik}}{G}, \quad G^{ik} = \begin{vmatrix} g_{ps} & g_{pt} \\ g_{rs} & g_{rt} \end{vmatrix}$$

and

$$g_{ik} = \frac{G_{ik}}{G'}, \quad G_{ik} = \begin{vmatrix} g^{ps} & g^{pt} \\ g^{rs} & g^{rt} \end{vmatrix},$$

where $G = \det[g_{ik}] = V^2$, $G' = \det[g^{ik}] = 1/G$ and (i, p, r), (k, s, t) are cyclic permutations of $(1, 2, 3)$. (In the orthogonal basis case these reduce to $g^{11} = 1/g_{11}$, $g^{22} = 1/g_{22}$, $g^{33} = 1/g_{33}$, with all other components zero.) Then, if we know covariant components a_k of a vector, we can calculate its contravariant components using $a^i = \sum_{k=1}^{3} g^{ik} a_k$ (this is called *raising the index*). Similarly, if we know contravariant components a^k, we can compute its covariant ones via $a_i = \sum_{k=1}^{3} g_{ik} a^k$ (*lowering the index*).