Tensor transformation under rotation about a coordinate axis and invariance of tensor equations

Math 1550 lecture notes, Prof. Anna Vainchtein

1 Tensor transformation under rotation about a coordinate axis

Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be an orthonormal basis and suppose a new orthonormal basis, $\{\bar{\mathbf{e}}_\mathbf{1}, \bar{\mathbf{e}}_\mathbf{2}, \bar{\mathbf{e}}_\mathbf{3}\}$ is obtained by rotating the old basis counterclockwise through an angle θ about $\mathbf{e_3}$, as shown in Fig. 1. Thus we have

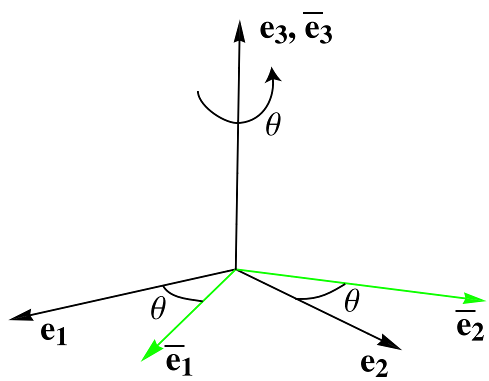

Figure 1: Rotation of the basis about $\mathbf{e_3}$ by an angle θ.

$$\bar{\mathbf{e}}_\mathbf{1} = \cos\theta \mathbf{e_1} + \sin\theta \mathbf{e_2}, \quad \bar{\mathbf{e}}_\mathbf{2} = -\sin\theta \mathbf{e_1} + \cos\theta \mathbf{e_2}, \quad \bar{\mathbf{e}}_\mathbf{3} = \mathbf{e_3}.$$

Recalling that $Q_{ij} = \bar{\mathbf{e}}_\mathbf{i} \cdot \mathbf{e_j}$, we obtain

$$\mathbf{Q} = [Q_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recall that components of a vector (first order tensor) $\mathbf{a}$ transform according to $\bar{a}_k = Q_{ik} a_k$ under the change of basis, i.e.

$$\begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1\cos\theta + a_2\sin\theta \\ -a_1\sin\theta + a_2\cos\theta \\ a_3 \end{bmatrix}.$$

Note that the scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is invariant under the rotation:

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2 + \bar{a}_3\bar{b}_3.$$

Indeed,

$$\begin{aligned} \bar{a}_1\bar{b}_1 + \bar{a}_2\bar{b}_2 + \bar{a}_3\bar{b}_3 &= (a_1\cos\theta + a_2\sin\theta)(b_1\cos\theta + b_2\sin\theta) \\ &+ (-a_1\sin\theta + a_2\cos\theta)(-b_1\sin\theta + b_2\cos\theta) + a_3b_3 \\ &= a_1b_1\cos^2\theta + a_2b_1\cos\theta\sin\theta + a_1b_2\cos\theta\sin\theta + a_2b_2\sin^2\theta \\ &+ a_1b_1\sin^2\theta - a_2b_1\cos\theta\sin\theta - a_1b_2\sin\theta\cos\theta + a_2b_2\cos^2\theta + a_3b_3 \\ &= a_1b_1 + a_2b_2 + a_3b_3. \end{aligned}$$

Of course, this is to be expected since a scalar priduct is a *scalar* and thus invariant under *any* change of coordinates. What about the vector product, $\mathbf{a}\times\mathbf{b}$? Its first and second components will change under the rotation but its third component is invariant, since $\bar{\mathbf{e}}_3 = \mathbf{e}_3$:

$$(\mathbf{a}\times\mathbf{b})\cdot\bar{\mathbf{e}}_3 = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}_3,$$

or

$$\bar{a}_1\bar{b}_2 - \bar{a}_2\bar{b}_1 = a_1b_2 - a_2b_1.$$

Indeed,

$$\begin{aligned} \bar{a}_1\bar{b}_2 - \bar{a}_2\bar{b}_1 &= (a_1\cos\theta + a_2\sin\theta)(-b_1\sin\theta + b_2\cos\theta) \\ &- (-a_1\sin\theta + a_2\cos\theta)(b_1\cos\theta + b_2\sin\theta) \\ &= -a_1b_1\sin\theta\cos\theta - a_2b_1\sin^2\theta + a_1b_2\cos^2\theta + a_2b_2\sin\theta\cos\theta \\ &+ a_1b_1\sin\theta\cos\theta - a_2b_1\cos^2\theta + a_1b_2\sin^2\theta - a_2b_2\sin\theta\cos\theta \\ &= a_1b_2 - a_2b_1. \end{aligned}$$

Consider now a second-order tensor $\mathbf{A}$ and recall that under the change of basis it must transform as

$$\bar{A}_{ij} = Q_{im}Q_{jn}A_{mn},$$

or, using matrix notation, $\bar{\mathbf{A}} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{\mathbf{T}}$. Thus,

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In particular, this yields

$$\bar{A}_{11} = Q_{1m}Q_{1n}A_{mn} = A_{11}\cos^2\theta + A_{21}\sin\theta\cos\theta + A_{12}\cos\theta\sin\theta + A_{22}\sin^2\theta.$$

Recalling that

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \sin\theta\cos\theta = \frac{\sin 2\theta}{2},$$

we then obtain

$$\bar{A}_{11} = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + \frac{A_{12}+A_{21}}{2}\sin 2\theta. \tag{1}$$

Similarly, we get (check!)

$$\bar{A}_{12} = \frac{A_{12}-A_{21}}{2} + \frac{A_{12}+A_{21}}{2}\cos 2\theta - \frac{A_{11}-A_{22}}{2}\sin 2\theta, \tag{2}$$

$$\bar{A}_{21} = -\frac{A_{12}-A_{21}}{2} + \frac{A_{12}+A_{21}}{2}\cos 2\theta - \frac{A_{11}-A_{22}}{2}\sin 2\theta, \tag{3}$$

$$\bar{A}_{22} = \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - \frac{A_{12}+A_{21}}{2}\sin 2\theta \tag{4}$$

and

$$\begin{aligned} &\bar{A}_{13} = A_{13}\cos\theta + A_{23}\sin\theta, \quad \bar{A}_{23} = -A_{13}\cos\theta + A_{23}\sin\theta, \\ &\bar{A}_{31} = A_{31}\cos\theta + A_{32}\sin\theta, \quad \bar{A}_{32} = -A_{31}\sin\theta + A_{32}\cos\theta, \quad \bar{A}_{33} = A_{33} \end{aligned} \tag{5}$$

In the special case when $\mathbf{A}$ is symmetruc, i.e. $A_{ij} = A_{ji}$, and in addition we have $A_{13} = A_{23} = 0$, the relations (1)-(5) reduce to the Mohr's circle relations used in engineering:

$$\begin{aligned} \bar{A}_{11} &= \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta, \\ \bar{A}_{22} &= \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - A_{12}\sin 2\theta, \\ \bar{A}_{12} &= -\frac{A_{11}-A_{22}}{2}\sin 2\theta + A_{12}\cos 2\theta; \end{aligned}$$

note that $\bar{A}_{21} = \bar{A}_{12}$, $\bar{A}_{13} = \bar{A}_{23} = 0$ and $\bar{A}_{33} = A_{33}$ in this case.

2 Invariance of tensor equations

Tensors change their components under coordinate transformations. But the physical laws involving tensors should be *invariant*, i.e. have the same form, under the change of coordinates such as rotation, reflection and translation with constant speed. This is ensured by our rule for how tensors must transform. For example, consider two inertial rectangular coordinate systems, K and $\bar{K}$, with orthonormal basis vectors $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ and $\{\bar{\mathbf{e}}_\mathbf{1}, \bar{\mathbf{e}}_\mathbf{2}, \bar{\mathbf{e}}_\mathbf{3}\}$, respectively, and suppose $\bar{K}$ is moving from K with constant translational velocity $\mathbf{v^0}$:

$$\bar{x}_i = Q_{ij} x_j + v_j^0 t, \quad Q_{ij} = \bar{\mathbf{e}}_\mathbf{i} \cdot \mathbf{e_j}.$$

In K Newton's second law is

$$F_j = \frac{d}{dt} (m v_j),$$

where F_j are components of the force $\mathbf{F}$ applied to an object of mass $m(t)$, and v_j are components of its velocity $\mathbf{v} = \mathbf{x}'(t)$, all measured in K. Under the coordinate change, $\bar{F}_i = Q_{ij} F_j$, so we have

$$\bar{F}_i = Q_{ij} F_j = \frac{d}{dt} (m Q_{ij} v_j) = \frac{d}{dt} \left[m \left(Q_{ij} v_j + v_j^0 \right) \right],$$

where we used the fact that $\mathbf{v^0}$ is constant. Since $\bar{v}_i = Q_{ij} v_j + v_j^0$ and $\bar{m} = m$, $\bar{t} = t$ (mass and time are scalars and thus invariant under the coordinate change), this implies

$$\bar{F}_i = \frac{d}{d\bar{t}} (\bar{m} \bar{v}_i),$$

i.e. Newton's second law has the same form in $\bar{K}$.