Tensors in generalized coordinate systems: components and direct notation

Math 1550 lecture notes, Prof. Anna Vainchtein

1 Vectors in generalized coordinates

Consider a generalized coordinate system (x^1, x^2, x^3) with the local basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. The basis is not necessarily orthogonal, let alone orthonormal. It comes along with its reciprocal basis $\{\mathbf{e^1}, \mathbf{e^2}, \mathbf{e^3}\}$. Recall that we can write a vector $\mathbf{a}$ in terms of either basis, using *contravariant* components a^i and *covariant* components a_i, respectively:

$$\mathbf{a} = a^1\mathbf{e_1} + a^2\mathbf{e_2} + a^3\mathbf{e_3} = a^i\mathbf{e_i}, \quad \mathbf{a} = a_1\mathbf{e^1} + a_2\mathbf{e^2} + a_3\mathbf{e^3} = a_i\mathbf{e^i}. \tag{1}$$

Recall also that we can find the covariant and contravariant components of the vector by taking dot products with the basis vectors and reciprocal basis vectors, respectively:

$$a_i = \mathbf{a}\cdot\mathbf{e_i}, \quad a^i = \mathbf{a}\cdot\mathbf{e^i} \tag{2}$$

Consider now another coordinate system $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, with the basis vectors

$$\mathbf{\bar{e}_i} = \alpha_i^p\mathbf{e_p}, \quad \alpha_i^p = \mathbf{\bar{e}_i}\cdot\mathbf{e^p}. \tag{3}$$

Recall that the reciprocal basis vectors then transform via the inverse transformation (see the first set of notes):

$$\mathbf{\bar{e}^i} = (\alpha^{-1})_p^i\mathbf{e^p}, \quad (\alpha^{-1})_p^i = \mathbf{e_p}\cdot\mathbf{\bar{e}^i} \tag{4}$$

Recall also that covariant and contravariant components of a vector (first order tensor) transform according to

$$\bar{a}_i = \alpha_i^p a_p, \quad \bar{a}^i = (\alpha^{-1})_p^i a^p. \tag{5}$$

Notice that that the transformation law for the covariant components involves the *direct* transformation matrix $\boldsymbol{\alpha}$, while the one for contravariant components has the *inverse* transformation matrix $\boldsymbol{\alpha}^{-1}$. As we have seen before, this is because $\bar{a}_i = \mathbf{a}\cdot\bar{\mathbf{e}}_\mathbf{i}$, where $\bar{\mathbf{e}}_\mathbf{i}$ is related to $\mathbf{e}_\mathbf{i}$ via the direct transformation, while in $\bar{a}^i = \mathbf{a}\cdot\bar{\mathbf{e}}^\mathbf{i}$, the reciprocal vector $\bar{\mathbf{e}}^\mathbf{i}$ transforms according to (4).

In particular, the new and old coordinates of the same point with position vector

$$\mathbf{r} = x^1\mathbf{e_1} + x^2\mathbf{e_2} + x^3\mathbf{e_3} = \bar{x}^1\bar{\mathbf{e}}_\mathbf{1} + \bar{x}^2\bar{\mathbf{e}}_\mathbf{2} + \bar{x}^3\bar{\mathbf{e}}_\mathbf{3}$$

are related by

$$\bar{x}^i = (\alpha^{-1})^i_p x^p, \quad x^p = \alpha^p_i \bar{x}^i,$$

and thus we can also represent the direct and inverse transformation matrices in terms of partial derivatives of the old and new coordinates with respect to one another:

$$\alpha^p_i = \frac{\partial x^p}{\partial \bar{x}^i}, \quad (\alpha^{-1})^i_p = \frac{\partial \bar{x}^i}{\partial x^p} \tag{6}$$

Using this, we can rewrite (5) as

$$\bar{a}_i = \frac{\partial x^p}{\partial \bar{x}^i} a_p, \quad \bar{a}^i = \frac{\partial \bar{x}^i}{\partial x^p} a^p \tag{7}$$

Finally, recall that covariant and contravariant components are *not* independent. They are related by

$$a^i = g^{ik} a_k, \quad a_i = g_{ik} a^k,$$

where we recall that $g_{ik} = \mathbf{e_i}\cdot\mathbf{e_k}$ and $g^{ik} = \mathbf{e^i}\cdot\mathbf{e^k}$ are covariant and contravariant components of the metric tensor. We called this raising or lowering the index via the metric tensor.

Remark. In the case of two rectangular coordinate systems with orthonormal bases that we considered earlier, we have $\mathbf{e^i} = \mathbf{e_i}$, $\bar{\mathbf{e}}^\mathbf{i} = \mathbf{e^i}$, $a_i = a^i$ and $\boldsymbol{\alpha}$ becomes an orthogonal matrix, $\boldsymbol{\alpha^{-1}} = \boldsymbol{\alpha^T}$. Thus, in this case we have $\alpha^j_i = Q_{ij}$ and $(\alpha^{-1})^j_i = Q_{ji}$, where $Q_{ij} = \bar{\mathbf{e}}_\mathbf{i}\cdot\mathbf{e_j}$ are components of an orthogonal matrix.

Example 1. Consider the vector

$$\mathbf{a} = x_1x_2\mathbf{i_1} + x_2x_3\mathbf{i_2} + x_1x_3\mathbf{i_3},$$

where $\{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}\}$ is the standard basis in the Cartesian coordinate system (x_1, x_2, x_3).

a) Find the covariant components of **a** in parabolic cylindrical coordinates (v, w, z) defined by

$$x_1 = \frac{v^2 - w^2}{2}, \quad x_2 = vw, \quad x_3 = z.$$

b) Express its contravariant components in terms of covariant ones.

Solution. a) The new basis is

$$\begin{aligned}
\bar{\mathbf{e}}_\mathbf{1} &= \frac{\partial x_1}{\partial u}\mathbf{i_1} + \frac{\partial x_2}{\partial u}\mathbf{i_2} + \frac{\partial x_3}{\partial u}\mathbf{i_3} = v\mathbf{i_1} + w\mathbf{i_2} \\
\bar{\mathbf{e}}_\mathbf{2} &= \frac{\partial x_1}{\partial v}\mathbf{i_1} + \frac{\partial x_2}{\partial v}\mathbf{i_2} + \frac{\partial x_3}{\partial v}\mathbf{i_3} = -w\mathbf{i_1} + v\mathbf{i_2} \\
\bar{\mathbf{e}}_\mathbf{3} &= \frac{\partial x_1}{\partial z}\mathbf{i_1} + \frac{\partial x_2}{\partial z}\mathbf{i_2} + \frac{\partial x_3}{\partial z}\mathbf{i_3} = \mathbf{i_3}
\end{aligned}$$

The transformation matrix is

$$[\alpha_i^j] = [\bar{\mathbf{e}}_\mathbf{i} \cdot \mathbf{i_j}] = \begin{bmatrix} v & w & 0 \\ -w & v & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The covariant components of **a** in the new basis are $\bar{a}_i = \alpha_i^j a_j$, or

$$\begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{bmatrix} = \begin{bmatrix} v & w & 0 \\ -w & v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(v^2 - w^2)vw \\ vwz \\ \frac{1}{2}(v^2 - w^2)z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(v^2 - w^2)v^2 w + vw^2 z \\ -\frac{1}{2}(v^2 - w^2)vw^2 + v^2 wz \\ \frac{1}{2}(v^2 - w^2)z \end{bmatrix}.$$

b) Note that the new basis is orthogonal, with $h_1 = |\bar{\mathbf{e}}_\mathbf{1}| = \sqrt{v^2 + w^2} = |\bar{\mathbf{e}}_\mathbf{2}| = h_2$ and $h_3 = |\bar{\mathbf{e}}_\mathbf{3}| = 1$. Thus the contravariant components of the metric tensor are $\bar{g}^{ij} = 0$ for $i \neq j$ and $\bar{g}^{11} = \bar{g}^{22} = \dfrac{1}{v^2 + w^2}$ and $\bar{g}^{33} = 1$. Therefore,

$$\bar{a}^1 = \bar{g}^{11}\bar{a}_1 = \frac{1}{v^2 + w^2}\bar{a}_1, \quad \bar{a}^1 = \bar{g}^{22}\bar{a}_2 = \frac{1}{v^2 + w^2}\bar{a}_2, \quad \bar{a}^3 = \bar{g}^{33}\bar{a}_3 = \bar{a}_3.$$

Example 2. Let $f(x^1, x^2, x^3)$ be a scalar field. If we change to new coordinates $\bar{x}^i = \bar{x}^i(x^1, x^2, x^3)$, we have, via the chain rule,

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^i},$$

which by the first equation in (6) yields

$$\frac{\partial f}{\partial \bar{x}^i} = \alpha_i^k \frac{\partial f}{\partial x^k}.$$

Thus, $v_i = \dfrac{\partial f}{\partial x^i}$ transform as *covariant* components of the vector (gradient of f)

$$\mathbf{v} = \frac{\partial f}{\partial x^i}\mathbf{e^i} = \nabla f,$$

where $\mathbf{e^i}$ are the reciprocal basis vectors associated with coordinates (x^1, x^2, x^3). Of course, the vector $\mathbf{v}$ also has *contravariant* components

$$v^i = g^{ik} v_k = g^{ik} \frac{\partial f}{\partial x^k}, \quad \mathbf{v} = g^{ik} \frac{\partial f}{\partial x^k}\mathbf{e_i}.$$

2 General higher order tensors

By analogy with above transformation laws for covariant and contravariant components of a vector, we can now introduce a more general definition of a second order tensor, no longer considering only rectangular coordinate systems:

Definition. A *second order tensor* in $\mathbb{R}^d$ is a quantity uniquely specified by d^2 numbers (its components). These components can be *covariant* (A_{ij}), *contravariant* (A^{ij}) or *mixed* ($A_{i}^{\cdot j}$, $A_{\cdot j}^{i}$), and they transform under the change of basis (3) according to

$$\bar{A}_{ij} = \alpha_i^p \alpha_j^q A_{pq} \tag{8}$$

$$\bar{A}^{ij} = (\alpha^{-1})_p^i (\alpha^{-1})_q^j A^{pq} \tag{9}$$

$$\bar{A}_i^{\cdot j} = \alpha_i^p (\alpha^{-1})_q^j A_p^{\cdot q} \tag{10}$$

$$\bar{A}_{\cdot j}^{i} = (\alpha^{-1})_p^i \alpha_j^q A_{\cdot q}^{p} \tag{11}$$

The little dot helps us denote the actual position of the index, e.g. in $A_i^{\cdot j}$ the index j comes second. Since $A_i^{\cdot j} \neq A_{\cdot i}^{j}$ in general, writing A_i^j is misleading, since we don't know which of the two is actually meant (Kronecker delta $\delta_i^j = \delta_j^i$ is an exception). Note that in the transformation laws the *covariant* indices always require *direct* transformation matrix $\boldsymbol{\alpha}$, while the contravariant indices come along with *inverse* transformation matrix $\boldsymbol{\alpha^{-1}}$, just like it was for the vector transformation laws (5). The indices in the transformation matrices are arranged so that the indices that are not summed over are in the same position as in the new component, while the summation is over the *opposite* indices. The only exception is Cartesian coordinates with orthonormal bases, where reciprocal vectors coincide with the regular ones, and the covariant, contravariant and mixed components coincide for each pair of indices.

Recalling that the transformation matrix can be represented as (6), we can also write (8)-(11) as

$$\begin{aligned} \bar{A}_{ij} &= \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}A_{pq} \\ \bar{A}^{ij} &= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}A^{pq} \\ \bar{A}_i^{\cdot j} &= \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial \bar{x}^j}{\partial x^q}A_p^{\cdot q} \\ \bar{A}^i_{\cdot j} &= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j}A^p_{\cdot q} \end{aligned} \tag{12}$$

This is how second order tensors are defined in some books.

Similar to vectors, the covariant, contravariant and mixed components of a second order tensor are related to one another via the metric tensor, which raises or lowers the corresponding indices. We have

$$\begin{aligned} A_{ij} &= g_{im}g_{jn}A^{mn} = g_{jn}A_i^{\cdot n} = g_{im}A^m_{\cdot j} \\ A^{ij} &= g^{im}g^{jn}A_{mn} = g^{im}A_m^{\cdot j} = g^{jn}A^i_{\cdot n} \\ A_i^{\cdot j} &= g^{jn}A_{in} = g_{im}A^{mj} \\ A^i_{\cdot j} &= g^{im}A_{mj} = g_{jn}A^{in} \end{aligned} \tag{13}$$

We can now easily generalize this to write transformation laws for higher order tensors. For example,

$$\begin{aligned} \bar{A}_{ijk} &= \alpha_i^p\alpha_j^q\alpha_k^r A_{pqr} \\ \bar{A}^{ijk} &= (\alpha^{-1})_p^i(\alpha^{-1})_q^j(\alpha^{-1})_r^k A^{pqr} \\ \bar{A}^{ij}_{\cdot\cdot k} &= (\alpha^{-1})_p^i(\alpha^{-1})_q^j\alpha_k^r A^{pq}_{\cdot\cdot r} \\ \bar{A}_{i\cdot k}^{\cdot j\cdot} &= \alpha_i^p(\alpha^{-1})_q^j\alpha_k^r A_{p\cdot r}^{\cdot q\cdot}, \end{aligned}$$

and so on. Once again, we define a third order tensor as an object whose various components transform according to these rules - and this is what we need to check to verify these are components of a tensor. The components of a third order tensor are again related by the metric tensor:

$$A_{ijk} = g_{im} A^{m}_{\cdot jk} = g_{im} g_{in} A^{mn}_{\cdot\cdot j} = g_{im} g_{jn} g_{kl} A^{mnl},$$

etc.

Exercise. Write the transformation laws for the components A_{ijkl}, A^{ijkl}, $A_{ij}^{\cdot\cdot kl}$ and $A_{i\cdot\cdot l}^{\cdot jk\cdot}$ of a fourth order tensor and the relations between these components.

Example 1. Contravariant components of a tensor $\mathbf{A}$ in the basis

$$\mathbf{e_1} = (0,1,1), \quad \mathbf{e_2} = (1,0,1), \quad \mathbf{e_3} = (1,1,1)$$

are

$$[A^{ij}] = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix}.$$

Find its mixed components $A^{i}_{\cdot j}$ and $A_{i}^{\cdot j}$ and covariant components A_{ij}.

Solution. We have $A^{i}_{\cdot j} = A^{im} g_{mj}$. The metric tensor is given by

$$[g_{mj}] = [\mathbf{e_m} \cdot \mathbf{e_j}] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

Therefore,

$$[A^{i}_{\cdot j}] = [A^{im}][g_{mj}] = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ 10 & 8 & 13 \\ -1 & 2 & 0 \end{bmatrix}.$$

Next,

$$[A_{i}^{\cdot j}] = [g_{im}][A^{mj}] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 10 & -1 \\ 3 & 8 & 2 \\ 2 & 13 & 0 \end{bmatrix}.$$

and finally,

$$[A_{ij}] = [A_i^{\cdot m}][g_{mj}] = \begin{bmatrix} 0 & 10 & -1 \\ 3 & 8 & 2 \\ 2 & 13 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 18 & 17 \\ 18 & 23 & 28 \\ 17 & 28 & 30 \end{bmatrix}.$$

Note that due to the symmetry of $[A^{ij}]$ and the metric tensor, the matrix $[A_{ij}]$ is also symmetric, while the matrices of the mixed components are not symmetric but are transposes of one another (not true in general).

Example 2. In a rectangular coordinate system with an orthonormal basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ the components of the second order tensor $\mathbf{A}$ are

$$[A_{ij}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 3 & 2 \end{bmatrix}$$

(Observe that $A^{ij} = A^i_{\cdot j} = A_i^{\cdot j} = A_{ij}$ in this case). Find the covariant components of this tensor in the coordinate system with the basis

$$\mathbf{\bar{e}_1} = \mathbf{e_1} + \mathbf{e_2}, \quad \mathbf{\bar{e}_2} = \mathbf{e_2} - \mathbf{e_3}, \quad \mathbf{\bar{e}_3} = \mathbf{e_1} + 2\mathbf{e_3}.$$

Solution. Recalling that $\mathbf{\bar{e}_i} = \alpha_i^k \mathbf{e_k}$, we have

$$\boldsymbol{\alpha} = [\alpha_i^k] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

(the rows are simply complonents of $\mathbf{\bar{e}_i}$ in the old basis). By (8), we have

$$[\bar{A}_{ij}] = \boldsymbol{\alpha}[A_{pq}]\boldsymbol{\alpha}^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 \\ -1 & -3 & 1 \\ 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 7 \\ -4 & -4 & 1 \\ 7 & 1 & 10 \end{bmatrix}.$$

Example 3. Show that $\dfrac{\partial v_i}{\partial x^k}$, where $v_i(x^1, x^2, x^3)$ are covariant components of a vector field, *do not* form a second order tensor.

Solution. By (7), $\bar{v}_i = \dfrac{\partial x^m}{\partial \bar{x}^i} v_m$, so

$$\frac{\partial \bar{v}_i}{\partial \bar{x}^k} = \frac{\partial}{\partial \bar{x}^k}\left(\frac{\partial x^m}{\partial \bar{x}^i} v_m\right) = \frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial v_m}{\partial \bar{x}^k} + \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^i} v_m = \frac{\partial x^m}{\partial \bar{x}^i}\frac{\partial x^n}{\partial \bar{x}^k}\frac{\partial v_m}{\partial x^n} + \boxed{\frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^i} v_m}$$

where we used chain rule to obtain the first term in the last equality.

Now, **without** the boxed second term, this *would* result in the transformation rule for covariant components $A_{ik} = \dfrac{\partial v_i}{\partial x^k}$ - see the first equality in (12). The problem is, this term only vanishes if $\dfrac{\partial x^m}{\partial \bar{x}^i} = \text{const}$, i.e. if we restrict ourselves to rectangular or oblique coordinate systems. In general, however, this term is nonzero, so $\dfrac{\partial v_i}{\partial x^k}$ are not components of a (general) second order tensor. Later we will show that adding a suitable quantity to $\dfrac{\partial v_i}{\partial x^k}$ fixes the problem and yields a second order tensor.

3 Metric tensor

We are now in position to show that the metric tensor is indeed a second order tensor, according to our definition. We need to check that its components transform according to the rules (8)-(11). Indeed, by (3) and the definition of metric tensor in the old and new bases,

$$\bar{g}_{ij} = \bar{\mathbf{e}}_\mathbf{i} \cdot \bar{\mathbf{e}}_\mathbf{j} = \alpha_i^m \mathbf{e}_\mathbf{m} \cdot \alpha_j^n \mathbf{e}_\mathbf{n} = \alpha_i^m \alpha_j^n (\mathbf{e}_\mathbf{m} \cdot \mathbf{e}_\mathbf{n}) = \alpha_i^m \alpha_j^n g_{mn},$$

confirming (8). Likewise, using (4), we obtain

$$\bar{g}^{ij} = \bar{\mathbf{e}}^\mathbf{i} \cdot \bar{\mathbf{e}}^\mathbf{j} = (\alpha^{-1})^i_m \mathbf{e}^\mathbf{m} \cdot (\alpha^{-1})^j_n \mathbf{e}_\mathbf{n} = (\alpha^{-1})^i_m (\alpha^{-1})^j_n (\mathbf{e}^\mathbf{m} \cdot \mathbf{e}_\mathbf{n}) = (\alpha^{-1})^i_m (\alpha^{-1})^j_n g^{mn},$$

so (9) also holds. Finally,

$$\bar{g}_i^{\cdot j} = \bar{\mathbf{e}}_\mathbf{i} \cdot \bar{\mathbf{e}}^\mathbf{j} = \alpha_i^m \mathbf{e}_\mathbf{m} \cdot (\alpha^{-1})^j_n \mathbf{e}^\mathbf{n} = \alpha_i^m (\alpha^{-1})^j_n (\mathbf{e}_\mathbf{m} \cdot \mathbf{e}^\mathbf{n}) = \alpha_i^m (\alpha^{-1})^j_n g_m^{\cdot n},$$

so (10) holds. We do not need to check (11) because $g^i_{\cdot j} = g_i^{\cdot j} = \delta_i^j$ (why?). So far, we have shown that all components transform according to the rules.

To show that these are the components of the *same* tensor, we also need to verify the relations (13). Note that $\mathbf{e_i} = (\mathbf{e}_i \cdot \mathbf{e_m})\mathbf{e^m} = g_{im}\mathbf{e^m}$, so that

$$g_{ij} = \mathbf{e_i} \cdot \mathbf{e_j} = g_{im}\mathbf{e^m} \cdot g_{jn}\mathbf{e^n} = g_{im}g_{jn}g^{mn},$$

verifying the first identity in (13). Also,

$$g_{ij} = \mathbf{e_i} \cdot \mathbf{e_j} = g_{im}\mathbf{e^m} \cdot \mathbf{e_j} = g_{im}g^{m}_{\cdot j}$$

verifies another identity in (13). In a similar way, we can show that the other identities also hold.

Exercise. Verify the other relations in (13) between the components of the metric tensor.

4 Tensor products

To write tensors using the *direct notation* instead of components, we need to introduce tensor products.

Let $\mathbf{a} = a_1\mathbf{i_1} + a_2\mathbf{i_2} + a_3\mathbf{i_3}$ and $\mathbf{b} = b_1\mathbf{i_1} + b_2\mathbf{i_2} + b_3\mathbf{i_3}$ be two vectors in $\mathbb{R}^3$. Here $\{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}\}$ is an orthonormal basis in a Cartesian coordinate system. Then the *tensor product* (also known as *dyadic product* and *outer product*) of the two vectors is the second order tensor

$$\mathbf{a} \otimes \mathbf{b}$$

whose components in the same basis are a_ib_j, i.e. it is represented by the matrix obtained by the matrix product of the two vectors:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}.$$

(We have shown in class that this is a Cartesian tensor.) For example, the tensor product $\mathbf{a} \otimes \mathbf{b}$ of $\mathbf{a} = \mathbf{i_1} + 2\mathbf{i_2} + 3\mathbf{i_3}$ and $\mathbf{b} = \mathbf{i_1} + \mathbf{i_2} + 2\mathbf{i_3}$ is represented by the matrix

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}.$$

Note that $\mathbf{b} \otimes \mathbf{a} = (\mathbf{a} \otimes \mathbf{b})^T$ - for instance, in the above example $\mathbf{b} \otimes \mathbf{a}$ has components $b_i a_j$ which form the matrix

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

that is transpose of $\mathbf{a} \otimes \mathbf{b}$.

In the same way we can define $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}$, a third order tensor with Cartesian components $a_i b_j c_k$, a fourth order tensor $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}$, etc.

An important property of the tensor product is that if we multiply it on the right by a vector, we get another vector given by

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}. \tag{14}$$

Indeed, in component notation we have (using summation convention - note the sum over j!)

$$(a_i b_j) c_j = (b_j c_j) a_i = (\mathbf{b} \cdot \mathbf{c}) a_i.$$

Similarly, if we multiply the tensor product on the left by a vector, we get

$$\mathbf{c} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b},$$

because $c_i(a_i b_j) = (c_i a_i) b_j = (\mathbf{c} \cdot \mathbf{a}) b_j$. Other properties of the tensor product are

$$(\gamma \mathbf{a}) \otimes \mathbf{b} = \mathbf{a} \otimes (\gamma \mathbf{b}) = \gamma (\mathbf{a} \otimes \mathbf{b}),$$

where γ is any scalar, and

$$\mathbf{a} \otimes (\mathbf{b} + \mathbf{c}) = \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{c}$$
$$(\mathbf{a} + \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \otimes \mathbf{c} + \mathbf{b} \otimes \mathbf{c}.$$

5 Tensors as linear combinations of tensor products of basis vectors

Recall that we used covariant and contravariant components of a vector to represent it in terms of basis vectors and reciprocal basis vectors, as in (1). We can do the same for higher order tensors, only now there are more types of components, and instead of basis vectors we will use their tensor products.

For example, using the contravariant components of a second order tensor $\mathbf{A}$, we can write it as

$$\mathbf{A} = A^{ij}\mathbf{e_i} \otimes \mathbf{e_j}. \tag{15}$$

Note that this is a double sum, so what we really mean is $\mathbf{A} = A^{11}\mathbf{e_1} \otimes \mathbf{e_1} + A^{12}\mathbf{e_1} \otimes \mathbf{e_2} + \cdots + A^{33}\mathbf{e_3} \otimes \mathbf{e_3}$. Now, for each i and j the tensor product $\mathbf{e_i} \otimes \mathbf{e_j}$ is itself a second order tensor, so we are writing our tensor $\mathbf{A}$ as a linear combination of such basis tensors. Observe also that

$$A^{ij} = \mathbf{e^i} \cdot \mathbf{A}\mathbf{e^j}, \tag{16}$$

which is the analog of the second identity in (2) for contravariant components of a vector. Indeed, by (15) (change the summation indices to m, n)

$$\mathbf{e^i} \cdot \mathbf{A}\mathbf{e^j} = \mathbf{e^i} \cdot A^{mn}(\mathbf{e_m} \otimes \mathbf{e_n})\mathbf{e^j}.$$

But by (14) we have

$$(\mathbf{e_m} \otimes \mathbf{e_n})\mathbf{e^j} = (\mathbf{e_n} \cdot \mathbf{e^j})\mathbf{e_m} = \delta_n^j \mathbf{e_m},$$

where in the last equality we used the property of the reciprocal bases. Thus,

$$\mathbf{e^i} \cdot \mathbf{A}\mathbf{e^j} = \mathbf{e^i} \cdot A^{mn}\delta_n^j \mathbf{e_m} = \mathbf{e^i} \cdot A^{mj}\mathbf{e_m} = A^{mj}(\mathbf{e^i} \cdot \mathbf{e_m}) = A^{mj}\delta_m^i = A^{ij},$$

yielding (16). To see that the representation (15), (16) is consistent with our definition of a second order tensor, we must show that the transformation law (9) holds for (16). The new contravariant components are

$$\bar{A}^{ij} = \bar{\mathbf{e}}^{\mathbf{i}} \cdot \mathbf{A}\bar{\mathbf{e}}^{\mathbf{j}}.$$

Substituting (4), we get

$$\bar{A}^{ij} = (\alpha^{-1})_p^i \mathbf{e^p} \cdot \mathbf{A}(\alpha^{-1})_q^j \mathbf{e^q} = (\alpha^{-1})_p^i (\alpha^{-1})_q^j (\mathbf{e^p} \cdot \mathbf{A}\mathbf{e^q}),$$

which by (16) yields (9). Thus, (16) indeed defines contravariant components of the second order tensor.

Now, instead of using the original basis vectors, we can represent the same tensor in terms of the tensor products of the reciprocal basis vectors and covariant components A_{ij}:

$$\mathbf{A} = A_{ij}\mathbf{e^i} \otimes \mathbf{e^j}, \quad A_{ij} = \mathbf{e_i} \cdot \mathbf{A}\mathbf{e_j}, \tag{17}$$

where the second identity is shown in the same way as (16) above (try it!) and is the analog of the first identity in (2) for covariant components of a vector. The new covariant components of $\mathbf{A}$ are

$$\bar{A}_{ij} = \bar{\mathbf{e}}_\mathbf{i} \cdot \mathbf{A}\bar{\mathbf{e}}_\mathbf{j}$$

Substituting (3), we get

$$\bar{A}_{ij} = \alpha_i^p \mathbf{e}_\mathbf{p} \cdot \mathbf{A}(\alpha_j^q \mathbf{e}_\mathbf{q}) = \alpha_i^p \alpha_j^q (\mathbf{e}_\mathbf{p} \cdot \mathbf{A}\mathbf{e}_\mathbf{q})$$

But $\mathbf{e}_\mathbf{p} \cdot \mathbf{A}\mathbf{e}_\mathbf{q} = A_{pq}$, so we get (8), confirming that the representation (17) is consistent with our earlier definition.

But why stop there? We can also mix the regular and reciprocal basis vectors and use the mixed components:

$$\mathbf{A} = A_i^{\cdot j} \mathbf{e}^\mathbf{i} \otimes \mathbf{e}_\mathbf{j}, \quad A_i^{\cdot j} = \mathbf{e}_\mathbf{i} \cdot \mathbf{A}\mathbf{e}^\mathbf{j} \tag{18}$$

and

$$\mathbf{A} = A_{\cdot j}^{i} \mathbf{e}_\mathbf{i} \otimes \mathbf{e}^\mathbf{j}, \quad A_{\cdot j}^{i} = \mathbf{e}^\mathbf{i} \cdot \mathbf{A}\mathbf{e}_\mathbf{j} \tag{19}$$

The new mixed components are

$$\bar{A}_i^{\cdot j} = \bar{\mathbf{e}}_\mathbf{i} \cdot \mathbf{A}\bar{\mathbf{e}}^\mathbf{j}, \quad \bar{A}_{\cdot j}^{i} = \bar{\mathbf{e}}^\mathbf{i} \cdot \mathbf{A}\bar{\mathbf{e}}_\mathbf{j}$$

Substituting (3) and (4) into the first of these, we get

$$\bar{A}_i^{\cdot j} = \alpha_i^p \mathbf{e}_\mathbf{i} \cdot \mathbf{A}(\alpha^{-1})_q^j \mathbf{e}^\mathbf{q},$$

and recalling the second identity in (18) we obtain (10), as expected.

Exercise. Use (3), (4) and (19) to obtain the transformation law (11) for the second type of mixed components.

Higher order tensors are can be written in terms of tensor products in a similar fashion. For instance, a third order tensor may be written as

$$\mathbf{A} = A_{ijk} \mathbf{e}^\mathbf{i} \otimes \mathbf{e}^\mathbf{j} \otimes \mathbf{e}^\mathbf{k} = A^{ijk} \mathbf{e}_\mathbf{i} \otimes \mathbf{e}_\mathbf{j} \otimes \mathbf{e}_\mathbf{k} = A_{\cdot\cdot k}^{ij} \mathbf{e}_\mathbf{i} \otimes \mathbf{e}_\mathbf{j} \otimes \mathbf{e}^\mathbf{k} = A_{i\cdot k}^{\cdot j \cdot} \mathbf{e}^\mathbf{i} \otimes \mathbf{e}_\mathbf{j} \otimes \mathbf{e}^\mathbf{k},$$

among the different possibilities. Again, the rule of thumb here is that summation is always done over the opposite indices. Note also that the order of the indices in the components and in the tensor products they multiply must be kept the same.

We can use the direct notation and the properties of tensor products to derive the relations between the different components of a tensor. For example, to see the first of the identities in (13), observe that

$$A_{ij} = \mathbf{e_i} \cdot \mathbf{A}\mathbf{e_j} = \mathbf{e_i} \cdot A^{mn}(\mathbf{e_m} \otimes \mathbf{e_n})\mathbf{e_j} = \mathbf{e_i} \cdot A^{mn}(\mathbf{e_n} \cdot \mathbf{e_j})\mathbf{e_m}$$

But $\mathbf{e_n} \cdot \mathbf{e_j} = g_{jn} = g_{nj}$, so that

$$A_{ij} = A^{mn} g_{jn} \mathbf{e_i} \cdot \mathbf{e_m} = g_{im} g_{jn} A^{mn}$$

because $\mathbf{e_i} \cdot \mathbf{e_m} = g_{im}$. To get the second identity in (13), just use a different representation of $\mathbf{A}$:

$$\begin{aligned} A_{ij} &= \mathbf{e_i} \cdot \mathbf{A}\mathbf{e_j} = \mathbf{e_i} \cdot A_m^{\cdot n}(\mathbf{e^m} \otimes \mathbf{e_n})\mathbf{e_j} = \mathbf{e_i} \cdot A_m^{\cdot n}\mathbf{e^m}(\mathbf{e_n} \cdot \mathbf{e_j}) = A_m^{\cdot n} g_{nj} \mathbf{e_i} \cdot \mathbf{e^m} \\ &= A_m^{\cdot n} g_{nj} \delta_i^m = A_i^{\cdot n} g_{nj} = g_{jn} A_i^{\cdot n} \end{aligned}$$

The other relations in (13) are derived the same way.