Pseudotensors

Math 1550 lecture notes, Prof. Anna Vainchtein

1 Proper and improper orthogonal transformations of bases

Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be an orthonormal basis in a Cartesian coordinate system and suppose we switch to another rectangular coordinate system with the orthonormal basis vectors $\{\bar{\mathbf{e}}_\mathbf{1}, \bar{\mathbf{e}}_\mathbf{2}, \bar{\mathbf{e}}_\mathbf{3}\}$. Recall that the two bases are related via an orthogonal matrix $\mathbf{Q}$ with components $Q_{ij} = \bar{\mathbf{e}}_\mathbf{i} \cdot \mathbf{e_j}$:

$$\bar{\mathbf{e}}_\mathbf{i} = Q_{ij}\mathbf{e_j}, \quad \mathbf{e_i} = Q_{ji}\bar{\mathbf{e}}_\mathbf{i}. \tag{1}$$

Let

$$\Delta = \det\mathbf{Q} \tag{2}$$

and recall that $\Delta = \pm 1$ because $\mathbf{Q}$ is orthogonal. If $\Delta = 1$, we say that the transformation is *proper orthogonal*; if $\Delta = -1$, it is an *improper orthogonal* transformation. Note that the handedness of the basis remains the same under a proper orthogonal transformation and *changes* under an improper one. Indeed,

$$\bar{V} = (\bar{\mathbf{e}}_\mathbf{1} \times \bar{\mathbf{e}}_\mathbf{2}) \cdot \bar{\mathbf{e}}_\mathbf{3} = (Q_{1m}\mathbf{e_m} \times Q_{2n}\mathbf{e_n}) \cdot Q_{3l}\mathbf{e_l} = Q_{1m}Q_{2n}Q_{3l}(\mathbf{e_m} \times \mathbf{e_n}) \cdot \mathbf{e_l}, \tag{3}$$

where the cross product is taken with respect to some underlying right-handed system, e.g. standard basis. Note that this is a triple sum over m, n and l. Now, observe that the terms in the above some where (m, n, l) is a cyclic (even) permutation of $(1, 2, 3)$ all multiply $V = (\mathbf{e_1} \times \mathbf{e_2}) \cdot \mathbf{e_3}$ because the scalar triple product is invariant under such permutations:

$$(\mathbf{e_1} \times \mathbf{e_2}) \cdot \mathbf{e_3} = (\mathbf{e_2} \times \mathbf{e_3}) \cdot \mathbf{e_1} = (\mathbf{e_3} \times \mathbf{e_1}) \cdot \mathbf{e_2}.$$

Meanwhile, terms where (m, n, l) is a non-cyclic (odd) permutations of $(1, 2, 3)$ multiply $-V$, e.g. for $(m, n, l) = (2, 1, 3)$ we have $(\mathbf{e_2} \times \mathbf{e_1}) \cdot \mathbf{e_3} = -(\mathbf{e_1} \times \mathbf{e_2}) \cdot \mathbf{e_3} = -V$. All other terms in the sum in (3) (where two or more of the three indices (m, n, l) are the same) are zero (why?). So (3) yields

$$\begin{aligned}\bar{V} = &(Q_{11}Q_{22}Q_{33} + Q_{13}Q_{21}Q_{32} + Q_{12}Q_{23}Q_{31} \\ &-Q_{12}Q_{21}Q_{33} - Q_{11}Q_{23}Q_{32} - Q_{13}Q_{22}Q_{31})V,\end{aligned}$$

or, recalling (2) and the definition of the determinant,

$$\bar{V} = \Delta V. \tag{4}$$

Hence $\bar{V}$ and V have the same sign (bases have the same handedness) if and only if $\Delta = 1$ (a proper orthogonal transformation). Otherwise, if $\Delta = -1$, a right-handed basis transforms into a left-handed one and vice versa.

Example 1. A counterclockwise rotation by the angle θ about x_3-axis is a proper orthogonal transformation. Indeed, in this case

$$[Q_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

which yields $\Delta = \cos^2\theta + \sin^2\theta = 1$. The handedness is preserved under such transformation (or any other pure rotation or combination thereof).

Example 2. Consider now a reflection in the (x_2, x_3) plane: $\bar{x}_1 = -x_1$, $\bar{x}_2 = x_2$, $\bar{x}_3 = x_3$. We have

$$[Q_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta = -1,$$

so the transformation is improper orthogonal. It is easy to see that such transformation reverses the handedness of the basis, as does any orthogonal transformation that involves an odd number of reflections.

2 Pseudotensors

Definition. A *pseudotensor* $\mathbf{A}$ of order n in $\mathbb{R}^3$ is a quantity with 3^n components which under orthogonal transformations of coordinate system transform as follows:

$$\bar{A}_{i_1 i_2 \ldots i_n} = \Delta Q_{i_1 j_1} Q_{i_2 j_2} \ldots Q_{i_n j_n} A_{j_1 j_2 \ldots j_n}.$$

In other words, pseudotensors transform the same way as ordinary tensors under *proper* orthogonal transformations but differently under the improper ones.

Example. Consider two vectors, $\mathbf{a}$ and $\mathbf{b}$, and let

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

be defined as their cross product *computed in the current basis.* Then we have

$$c_i = a_j b_k - a_k b_j, \quad \bar{c}_i = \bar{a}_j \bar{b}_k - \bar{a}_k \bar{b}_j, \tag{5}$$

where for any i, the indices j and k are selected so that (i, j, k) is the cyclic permutation of $(1, 2, 3)$.

Question: does $\mathbf{c}$ so defined transform as a Cartesian tensor, i.e. do we have $\bar{c}_i = Q_{ik} c_k$ under the change of basis (1)? Let's check:

$$\bar{c}_i = \bar{a}_j \bar{b}_k - \bar{a}_k \bar{b}_j = Q_{jm} a_m Q_{kn} b_n - Q_{kn} a_n Q_{jm} b_m = Q_{jm} Q_{kn} (a_m b_n - a_n b_m).$$

Interchanging the summation indices m and n in the second term ($-Q_{jm} Q_{kn} a_n b_m = -Q_{jn} Q_{km} a_m b_n$), we get

$$\bar{c}_i = (Q_{jm} Q_{kn} - Q_{jn} Q_{km}) a_m b_n. \tag{6}$$

Now, observe that the terms with $m = n$ in the above sum are zero, so only terms with $m \neq n$ remain. Now, if $m = 1$ and $n = 2$, say, we get $(Q_{j1} Q_{k2} - Q_{j2} Q_{k1}) a_1 b_2$, while $m = 2$ and $n = 1$ yields $(Q_{j2} Q_{k1} - Q_{j1} Q_{k2}) a_2 b_1$, so that the two terms combined give us

$$(Q_{j1} Q_{k2} - Q_{j2} Q_{k1})(a_1 b_2 - a_2 b_1).$$

Similarly, the terms with $(m, n) = (2, 3)$ and $(m, n) = (3, 2)$ yield

$$(Q_{j2} Q_{k3} - Q_{j3} Q_{k2})(a_2 b_3 - a_3 b_2)$$

and the terms with $(m, n) = (3, 1)$ and $(m, n) = (1, 3)$ yield

$$(Q_{j3} Q_{k1} - Q_{j1} Q_{k3})(a_3 b_1 - a_1 b_3).$$

Combining these, we can see that (6) becomes

$$\begin{aligned} \bar{c}_i = {} & (Q_{j1} Q_{k2} - Q_{j2} Q_{k1})(a_1 b_2 - a_2 b_1) \\ & + (Q_{j2} Q_{k3} - Q_{j3} Q_{k2})(a_2 b_3 - a_3 b_2) \\ & + (Q_{j3} Q_{k1} - Q_{j1} Q_{k3})(a_3 b_1 - a_1 b_3), \end{aligned} \tag{7}$$

where we recall that (i, j, k) is the cyclic permutation of $(1, 2, 3)$.

We now claim that

$$Q_{jm}Q_{kn} - Q_{jn}Q_{km} = \Delta Q_{il}, \tag{8}$$

where (i, j, k) and (m, n, l) are both cyclic permutations of $(1, 2, 3)$. To see this, note that by (1),

$$\bar{\mathbf{e}}_{\mathbf{j}} \times \bar{\mathbf{e}}_{\mathbf{k}} = Q_{jm}Q_{kn}\mathbf{e}_{\mathbf{m}} \times \mathbf{e}_{\mathbf{n}},$$

where both cross products are taken with respect to the same underlying right-handed basis, e.g. standard basis. Then

$$(\bar{\mathbf{e}}_{\mathbf{j}} \times \bar{\mathbf{e}}_{\mathbf{k}}) \cdot \mathbf{e}_{\mathbf{l}} = Q_{jm}Q_{kn}(\mathbf{e}_{\mathbf{m}} \times \mathbf{e}_{\mathbf{n}}) \cdot \mathbf{e}_{\mathbf{l}}.$$

As in the earlier calculation, we note that $(\mathbf{e}_{\mathbf{m}} \times \mathbf{e}_{\mathbf{n}}) \cdot \mathbf{e}_{\mathbf{l}}) = (\mathbf{e}_{\mathbf{1}} \times \mathbf{e}_{\mathbf{2}}) \cdot \mathbf{e}_{\mathbf{3}} = V$ if (m, n, l) is a cyclic permutation of $(1, 2, 3)$, $(\mathbf{e}_{\mathbf{m}} \times \mathbf{e}_{\mathbf{n}}) \cdot \mathbf{e}_{\mathbf{l}} = -V$ if it is non-cyclic and zero otherwise. This yields

$$(\bar{\mathbf{e}}_{\mathbf{j}} \times \bar{\mathbf{e}}_{\mathbf{k}}) \cdot \mathbf{e}_{\mathbf{l}} = (Q_{jm}Q_{kn} - Q_{jn}Q_{km})V, \tag{9}$$

where (m, n, l) is a cyclic permutation of $(1, 2, 3)$. But (i, j, k) is also a cyclic permutation of $(1, 2, 3)$, so we have

$$\bar{\mathbf{e}}_{\mathbf{j}} \times \bar{\mathbf{e}}_{\mathbf{k}} = \bar{V}\bar{\mathbf{e}}_{\mathbf{i}},$$

and hence

$$(\bar{\mathbf{e}}_{\mathbf{j}} \times \bar{\mathbf{e}}_{\mathbf{k}}) \cdot \mathbf{e}_{\mathbf{l}} = \bar{V}(\bar{\mathbf{e}}_{\mathbf{i}} \cdot \mathbf{e}_{\mathbf{l}}) = \bar{V}Q_{il}. \tag{10}$$

Combining (9) and (10), we get

$$\bar{V}Q_{il} = V(Q_{jm}Q_{kn} - Q_{jn}Q_{km}),$$

which by (4) yields (8).

Now observe that (8) with $(m, n) = (1, 2)$ (and thus $l = 3$) yields $Q_{j1}Q_{k2} - Q_{j2}Q_{k1} = \Delta Q_{i3}$. Similarly, with $(m, n) = (2, 3)$ we get $Q_{j2}Q_{k3} - Q_{j3}Q_{k2} = \Delta Q_{i1}$ and with $(m, n) = (3, 1)$ we have $Q_{j3}Q_{k1} - Q_{j1}Q_{k3} = \Delta Q_{i2}$. Substituting these in (7), we get

$$\bar{c}_i = \Delta\bigg(Q_{i3}(a_1b_2 - a_2b_1) + Q_{i1}(a_2b_3 - a_3b_3) + Q_{i2}(a_3b_1 - a_1b_3)\bigg) = \Delta Q_{il}(a_mb_n - a_nb_m),$$

where again (m, n, l) is a cyclic permutation of $(1, 2, 3)$. But this implies that

$$\bar{c}_i = \Delta Q_{il} c_l,$$

so that $\mathbf{c}$ defined as the cross product in the current basis is indeed a pseudotensor of first order (a *pseudovector*, or *axial vector*) rather than a regular vector. It transforms as a regular vector under proper orthogonal transformations ($\Delta = 1$), $\bar{c}_i = Q_{il} c_l$, but not under improper ones ($\Delta = -1$) where we have $\bar{c}_i = -Q_{il} c_l$.

Note that if we just compute $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ in some basis and then consider the transformation of the resulting quantity under a change to another basis (*without* recomputing the cross product with respect to the new basis), then of course the quantity will transform as a regular vector, $\bar{c}_i = Q_{il} c_l$. But the point is that the new components will *not* be components of the cross product $\mathbf{a} \times \mathbf{b}$ as computed *in the new basis* unless $\mathbf{Q}$ is proper orthogonal. So it is really *the cross product operation* that gives us a pseudovector because we use the convention (right hand rule in a right-handed basis and left hand rule in a left-handed one). The convention allows us to compute the cross product the same way,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

in *any* orthonormal basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, regardless of its handedness, but the price we pay is that it gives us a pseudovector if we compute the cross product in the current basis.

Remark. In general bases, the cross product is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix} \sqrt{G} = \begin{vmatrix} \mathbf{e^1} & \mathbf{e^2} & \mathbf{e^3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \frac{1}{\sqrt{G}},$$

where $G = V^2 = \det[g_{ik}]$ is the determinant of the metric tensor. In the orthonormal case $G = 1$ and the covariant and contravariant components coincide, as do regular and reciprocal basis vectors.

3 Properties of pseudotensors

1. The sum of two pseudotensors of order n is a pseudotensor of order n.

   Indeed, if $C_{i_1\dots i_n} = A_{i_1\dots i_n} + B_{i_1\dots i_n}$ and

   $$\bar{A}_{i_1\dots i_n} = \Delta Q_{i_1j_1}\dots Q_{i_nj_n}A_{j_1\dots j_n}, \quad \bar{B}_{i_1\dots i_n} = \Delta Q_{i_1j_1}\dots Q_{i_nj_n}B_{j_1\dots j_n},$$

   then $\bar{C}_{i_1\dots i_n} = \Delta Q_{i_1j_1}\dots Q_{i_nj_n}C_{j_1\dots j_n}$, so $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is a pseudotensor.

2. The product of a pseudotensor of order m and a regular tensor of order n is a pseudotensor of order $m+n$.

   Let $\mathbf{A}$ be a pseudotensor of order m and $\mathbf{B}$ be a regular tensor of order n. Then we have $\bar{A}_{i_1\dots i_m} = \Delta Q_{i_1j_1}\dots Q_{i_mj_m}A_{j_1\dots j_m}$ and $\bar{B}_{k_1\dots k_n} = Q_{k_1l_1}\dots Q_{k_nl_n}B_{l_1\dots l_n}$. Thus for $C_{i_1\dots i_mk_1\dots k_n} = A_{i_1\dots i_m}B_{k_1\dots k_n}$ we have $\bar{C}_{i_1\dots i_mk_1\dots k_n} = \Delta Q_{i_1j_1}\dots Q_{i_mj_m}Q_{k_1l_1}\dots Q_{k_nl_n}C_{j_1\dots j_ml_1\dots l_n}$, so $\mathbf{C}$ is a pseudotensor of order $m+n$.

3. The product of two pseudotensors of orders m and n is an ordinary (Cartesian) tensor of order $m+n$.

   Proved the same way as above, except now we get Δ for both $\mathbf{A}$ and $\mathbf{B}$, so we obtain $\Delta^2 = 1$ in the transformation law for $\mathbf{C}$.

4. Contraction of a pseudotensor of order $n \geq 2$ gives a pseudotensor of order $n-2$.

   Indeed, if $\bar{A}_{i_1i_2\dots i_n} = \Delta Q_{i_1j_1}Q_{i_2j_2}\dots Q_{i_nj_n}A_{j_1j_2\dots j_n}$, then its contraction in the kth and mth index (replace i_m by i_k and sum over these) satisfies

   $$\bar{A}_{i_1\dots i_k\dots i_k\dots i_n} = \Delta Q_{i_1j_1}\dots Q_{i_kj_k}\dots Q_{i_kj_m}\dots Q_{i_nj_n}A_{j_1\dots j_k\dots j_m\dots j_n}.$$

   But since $Q_{i_kj_k}Q_{i_kj_m} = \delta_{j_kj_m}$ this yields

   $$\bar{A}_{i_1\dots i_k\dots i_k\dots i_n} =$$
   $$\Delta Q_{i_1j_1}\dots Q_{i_{k-1}j_{k-1}}Q_{i_{k+1}j_{k+1}}\dots Q_{i_{m-1}j_{m-1}}Q_{i_{m+1}j_{m+1}}\dots Q_{i_nj_n}A_{j_1\dots j_k\dots j_k\dots j_n},$$

   so $A_{i_1\dots i_k\dots i_k\dots i_n}$ are components of a pseudotensor of order $n-2$.

4 Levi-Civita symbol

Consider

$$\varepsilon_{ijk} = (\mathbf{e_i} \times \mathbf{e_j}) \cdot \mathbf{e_k}, \tag{11}$$

where **the cross product is taken with respect to the orthonormal basis** $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. Then we have

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } (i,j,k) \text{ is a cyclic (even) permutation of } (1,2,3) \\ -1, & \text{if } (i,j,k) \text{ is a non-cyclic (odd) permutation of } (1,2,3) \\ 0, & \text{otherwise, i.e. whwen any two indices coincide} \end{cases} \tag{12}$$

Note that there are *only six nonzero components*:

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \quad \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$$

and observe that for any matrix $\mathbf{A}$, $\det\mathbf{A} = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$ (show this as an exercise). Observe also that

$$\varepsilon_{mnk} = \varepsilon_{nkm} = \varepsilon_{kmn}, \tag{13}$$

i.e. the components are invariant under the cyclic permutation of the indices.

We claim that ε_{ijk} is a *third order pseudotensor*. Indeed, in a new basis $\{\bar{\mathbf{e}}_\mathbf{1}, \bar{\mathbf{e}}_\mathbf{2}, \bar{\mathbf{e}}_\mathbf{3}\}$ we have

$$\bar{\varepsilon}_{ijk} = (\bar{\mathbf{e}}_\mathbf{i} \bar{\times} \bar{\mathbf{e}}_\mathbf{j}) \cdot \bar{\mathbf{e}}_\mathbf{k},$$

where the bar above the cross product sign indicates that we are taking the cross product **with respect to the new basis**. Recalling (1), we have

$$\bar{\varepsilon}_{ijk} = (Q_{im}\mathbf{e_m} \bar{\times} Q_{jn}\mathbf{e_n}) \cdot Q_{kl}\mathbf{e_l} = Q_{im}Q_{jn}Q_{kl}(\mathbf{e_m} \bar{\times} \mathbf{e_n}) \cdot \mathbf{e_l},$$

where again we are taking the cross product in the new coordinate system. But we know that the cross products taken in the new and old systems coincide, $(\mathbf{e_m} \bar{\times} \mathbf{e_n}) \cdot \mathbf{e_l} = (\mathbf{e_m} \times \mathbf{e_n}) \cdot \mathbf{e_l}$ if the two bases have the same handedness (and thus $\mathbf{Q}$ is a proper orthogonal transformation, $\Delta = 1$) and have opposite signs, $(\mathbf{e_m} \bar{\times} \mathbf{e_n}) \cdot \mathbf{e_l} = -(\mathbf{e_m} \times \mathbf{e_n}) \cdot \mathbf{e_l}$ if they have different handedness and thus the transformation is improper ($\Delta = -1$). This means that

$$(\mathbf{e_m} \bar{\times} \mathbf{e_n}) \cdot \mathbf{e_l} = \Delta(\mathbf{e_m} \times \mathbf{e_n}) \cdot \mathbf{e_l} = \Delta\varepsilon_{mnl},$$

and thus we have

$$\bar{\varepsilon}_{ijk} = \Delta Q_{im}Q_{jn}Q_{kl}\varepsilon_{mnl},$$

proving that ε_{ijk} is a third order pseudotensor.

Observe that if $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, we can now write

$$c_i = \varepsilon_{ijk} a_j b_k \tag{14}$$

in any Cartesian coordinate system. Indeed,

$$c_1 = \varepsilon_{1jk} a_j b_k = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2,$$

where we have used the fact that by (12) the only nonzero components ε_{1jk} are $\varepsilon_{123} = 1$ and $\varepsilon_{132} = -1$. Comparing this to the first equality in (5) with $(i, j, k) = (1, 2, 3)$, we see that this is indeed the first component of the cross product. Similarly, one can show that (14) yields $c_2 = a_3 b_1 - a_1 b_3$ and $c_3 = a_1 b_2 - a_2 b_1$, which are the second and third components of the cross product according to the first equality in (5) taken with $(i, j, k) = (2, 3, 1)$ and $(i, j, k) = (3, 1, 2)$, respectively.

Recalling the properties of pseudotensors, one can see that if ϕ is a scalar, then $\varepsilon_{ijk}\phi$ are components of a third order pseudotensor, and if T_{ij} are components of a second order tensor, then $\varepsilon_{ijk} T_{jk}$ are components of a first order pseudotensor, or pseudovector (obtained by the double contraction of a fifth order pseudotensor). For example, $T_{jk} = a_j b_k$ are components of a regular Cartensian tensor, ans so $c_i = \varepsilon_{ijk} T_{jk} = \varepsilon_{ijk} a_j b_k$ are components of a pseudovector. If T_{ijk} are components of a third order tensor, then $\varepsilon_{ijk} T_{ijk}$ is a *pseudoscalar* (meaning that it changes sign as we change from a right handed system to a left handed or vice versa). For example, $T_{ijk} = a_i b_j c_k$, we get the pseudoscalar

$$d = \varepsilon_{ijk} T_{ijk} = \varepsilon_{ijk} a_i b_j c_k = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Meanwhile, $\varepsilon_{ijk}\varepsilon_{mnl}$ is a sixth order regular Cartesian tensor, which satisfies the following identity:

$$\varepsilon_{ijk}\varepsilon_{mnl} = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{il} \\ \delta_{jm} & \delta_{jn} & \delta_{jl} \\ \delta_{km} & \delta_{kn} & \delta_{kl} \end{vmatrix}.$$

In particular, expanding this determinant about the third row and setting $l = k$ (which means summing over k), we obtain the fourth order tensor

$$\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{jm}\delta_{in}. \tag{15}$$

Indeed, the expansion gives us

$$\varepsilon_{ijk}\varepsilon_{mnl} = \delta_{kl}\begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} - \delta_{kn}\begin{vmatrix} \delta_{im} & \delta_{il} \\ \delta_{jm} & \delta_{jl} \end{vmatrix} + \delta_{km}\begin{vmatrix} \delta_{in} & \delta_{il} \\ \delta_{jn} & \delta_{jl} \end{vmatrix}.$$

Setting $l = k$, we get

$$\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{kk}\begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} - \delta_{kn}\begin{vmatrix} \delta_{im} & \delta_{ik} \\ \delta_{jm} & \delta_{jk} \end{vmatrix} + \delta_{km}\begin{vmatrix} \delta_{in} & \delta_{ik} \\ \delta_{jn} & \delta_{jk} \end{vmatrix}.$$

But $\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$, and due to the Kronecker Deltas in front of the determinants in the second and third terms, we can simplify this further as

$$\varepsilon_{ijk}\varepsilon_{mnk} = 3\begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} - \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} + \begin{vmatrix} \delta_{in} & \delta_{im} \\ \delta_{jn} & \delta_{jm} \end{vmatrix}$$

Observing that interchaging the columns in the last determinant changes its sign to the opposite, we get

$$\varepsilon_{ijk}\varepsilon_{mnk} = 2\begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} - \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} = \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix},$$

which yields (15).

Contracting (15) further, we obtain the second order tensor

$$\varepsilon_{ink}\varepsilon_{mnk} = 2\delta_{im}.$$

Indeed, setting $j = n$ in (15) (and thus summing over n), we get $\varepsilon_{ink}\varepsilon_{mnk} = \delta_{im}\delta_{nn} - \delta_{nm}\delta_{in}$. But $\delta_{nn} = 3$ (see above), so we get $\varepsilon_{ink}\varepsilon_{mnk} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$.

In particular, we can use (15) to prove

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Note that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a regular Cartesian vector (not a pseudovector) because we apply the cross product twice. To prove the identity, let $\mathbf{d} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Then by (14)

$$d_i = \varepsilon_{ijk}a_j(\mathbf{b} \times \mathbf{c})_k = \varepsilon_{ijk}a_j\varepsilon_{kmn}b_m c_n$$

But by (13) $\varepsilon_{kmn} = \varepsilon_{mnk}$, so that

$d_i = \varepsilon_{ijk}\varepsilon_{mnk} a_j b_m c_n = (\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in}) a_j b_m c_n = \delta_{im}\delta_{jn} a_j b_m c_n - \delta_{jm}\delta_{in} a_j b_m c_n$,

where we used (15) to get the first equality. Hence

$$d_i = a_n b_i c_n - a_m b_m c_i = b_i(a_n c_n) - c_i(a_m b_m) = b_i(\mathbf{a}\cdot\mathbf{c}) - c_i(\mathbf{a}\cdot\mathbf{b}).$$

Thus,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{d} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}),$$

proving the identity.

5 Antisymmetric second order tensors and pseudovectors

Let A_{ij} be components of an antisymmetric second order tensor, i.e. $A_{ji} = -A_{ij}$. The matrix representation of the tensor is

$$[A_{ij}] = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix},$$

so that there are only six nonzero components. Now observe that we can write

$$A_{ij} = \varepsilon_{ijk} a_k,$$

or

$$[A_{ij}] = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}.$$

Indeed, $A_{23} = \varepsilon_{231} a_1 = a_1$, $A_{12} = \varepsilon_{123} a_3 = a_3$, $A_{31} = \varepsilon_{312} a_2 = a_2$, while $A_{32} = \varepsilon_{321} a_1 = -a_1$, $A_{21} = \varepsilon_{213} a_3 = -a_3$ and $A_{13} = \varepsilon_{132} a_2 = -a_2$. The diagonal components are all zero because $\varepsilon_{iik} = 0$ (no sum). It follows that

$$a_i = \frac{1}{2}\varepsilon_{ijk} A_{jk}. \tag{16}$$

Indeed, $A_1 = \frac{1}{2}(\varepsilon_{123} A_{23} + \varepsilon_{132} A_{32}) = \frac{1}{2}(A_{23} - A_{32})$, $a_2 = \frac{1}{2}(\varepsilon_{231} A_{31} + \varepsilon_{213} A_{13}) = \frac{1}{2}(A_{31} - A_{13})$ and $a_3 = \frac{1}{2}(\varepsilon_{312} A_{12} + \varepsilon_{321} A_{21}) = \frac{1}{2}(A_{12} - A_{21})$. We write

$$\mathbf{a} = \frac{1}{2}\mathbf{A}_\times = \mathbf{vec}\mathbf{A}.$$

Since **a** is obtained by a double contraction of a second order tensor and a third order pseudotensor, the result must be a pseudovector (recall the properties of pseudotensors). And indeed, under a change of variables,

$$\bar{a}_i = \frac{1}{2}\bar{\varepsilon}_{ijk}\bar{A}_{jk} = \frac{1}{2}\Delta Q_{im}Q_{jn}Q_{kl}\varepsilon_{mnl}Q_{jp}Q_{kq}A_{pq}.$$

But $Q_{jn}Q_{jp} = \delta_{np}$ and $Q_{kl}Q_{kq} = \delta_{lq}$, so we have

$$\bar{a}_i = \frac{1}{2}\Delta Q_{im}\varepsilon_{mnl}\delta_{np}\delta_{lq}A_{pq} = \frac{1}{2}\Delta Q_{im}\varepsilon_{mnl}A_{nl} = \Delta Q_{im}a_m,$$

since by (16) $\varepsilon_{mnl}A_{nl} = 2a_m$. So **a**, whose components are related to the components of the antisymmetric tensor **A** via (16), transforms as a pseudovector, or an axial vector.