A NOTE ON TIME-REVERSIBILITY OF MULTIVARIATE LINEAR PROCESSES

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SUMMARY

We solve an important open problem by deriving some readily verifiable necessary and sufficient conditions for a multivariate non-Gaussian linear process to be time-reversible, under two sets of regularity conditions on the contemporaneous dependence structure of the innovations. One set of regularity conditions concerns the case of independent-component innovations, in which case a multivariate non-Gaussian linear process is time-reversible if and only if the coefficients consist of essentially symmetric columns with column-specific origins of symmetry or symmetric pairs of columns with pair-specific origins of symmetry. On the other hand, for dependent-component innovations plus other regularity conditions, a multivariate non-Gaussian linear process is time-reversible if and only if the coefficients are essentially symmetric about some origin.

Some Key Words: Cumulants; Distributional equivalence; Non-Gaussian distribution; Time series; t-distribution; Symmetry.

1 INTRODUCTION & LITERATURE REVIEW

Symmetry is all pervasive in art and science. Dawid (1988) has emphasised its role in statistics. In his discussion of the paper, Tong (*op. cit.*, p. 30) has identified time-reversibility as an additional symmetry for stationary time series, stationarity itself being an expression of symmetry. The stochastic process X is said to be time-reversible if its ensemble properties are invariant to the direction of time. In other words, X is time-reversible if for any finite set of epochs $t_1, t_2, \cdots, t_n$, the probability distribution of $\{X(t_1), X(t_2), \cdots, X(t_n)\}$ is identical to that of $\{X(-t_1), X(-t_2), \cdots, X(-t_n)\}$. Time-reversibility often induces simpler probabilistic structure, see Keilson (1979). In statistics, Lawrance (1991) has studied the issue of directionality in time series. Breidt and Davis (1992) has referred to the implications of time-reversibility to de-convolution. Diks *et al.* (1995) has exploited time-reversibility to develop a criterion for discriminating time series. Economists and financial engineers have addressed the asymmetry of business cycles via the notion of time-reversibility. (See, e.g. Ramsey and Rothman, 1995.) Tong and Zhang (2005) has discussed the role of time-reversibility in the statistical inference of time series models. However, almost all the discussions in the statistical literature to-date (Tong and Zhang (2005) being a rare exception) are restricted to the case of univariate time series, leaving the more important but also substantially more challenging case of multivariate time series virtually untouched.

Consider a p-dimensional linear process X driven by a q-dimensional independent and identically distributed innovation sequence ϵ defined by the equation:

$$X(t) \quad = \quad \sum_{j=-\infty}^{\infty} A(j)\epsilon(t-j), \quad t = \cdots, -2, -1, 0, 1, 2, \cdots, \tag{1}$$

where the coefficients $A(j)$ are assumed to be square-summable such that $\sum_j \|A(j)\|^2 < \infty$. Here for any matrix $A = (a_{ij})$, its square norm is defined as $\|A\|^2 = \sum_{i,j} a_{ij}^2$. The square summability condition of the innovations ensures that the linear process has finite second moments. Without loss of generality, the innovations are assumed to be of zero mean. To rule out singularity in the probability distribution, we further impose the regularity conditions that the innovations have a positive-definite covariance matrix and the transfer function (Fourier transform of the coefficients)

$\hat{A}(\omega) = \sum A(j)\exp(-ij\omega)$ is of full-rank almost everywhere on the interval $(-\pi, \pi]$ where $i = \sqrt{-1}$. While q can be greater than p, such a model is generally non-invertible, i.e. the innovations cannot be determined even if the entire X is known. Hence, we shall consider only the case when $p \geq q$.

For a Gaussian linear process, it is well known (e.g. Tong and Zhang, 2005) that it is time-reversible if and only if its autocovariances are all symmetric matrices. In particular, univariate Gaussian processes are always time-reversible. For univariate non-Gaussian linear processes, time-reversibility requires a more restrictive necessary and sufficient condition on the coefficients:

(NSC) There exist an integer m and a scalar b such that, for all t, (i) $A(t) = A(-t+m)b$ and (ii) $b\epsilon(t)$ is distributionally equivalent to $\epsilon(t)$.

See Cheng (1999) and Tong and Zhang (2005). Indeed, the stronger condition owes to the fact that the linear representation of a univariate non-Gaussian linear process is essentially unique while this is generally not the case for a Gaussian process; see Findley (1986, 1990) and Cheng (1992). Specifically, if the univariate non-Gaussian X admits another representation, say,

$$X(t) \quad = \quad \sum_{j=-\infty}^{\infty} A'(j)\epsilon'(t-j), \quad t = \cdots, -2, -1, 0, 1, 2, \cdots, \tag{2}$$

where ϵ' is a sequence of independent and identically distributed innovations, then there exist b and an integer m such that, for all t,

$$\epsilon'(t-m) = b\epsilon(t), \quad \text{and} \quad A(t-m) = A'(t)b. \tag{3}$$

See Rosenblatt (2000) for a review of statistical applications with non-Gaussian linear processes.

The situation with multivariate non-Gaussian linear processes is less clear and the problem of necessary and sufficient conditions for time-reversibility has remained open. Tong and Zhang (2005) obtained a necessary and sufficient condition for the case with arbitrary $p \geq 1$ but $q = 1$, i.e. univariate innovation sequence. They also derived a sufficient condition for time-reversibility in the general case but showed by a counter-example that their condition is not necessary. The Tong-Zhang condition requires that

(TZ) there exist an integer m and a matrix B such that, for all t, (i) $A(t) = A(-t+m)B$ and (ii) $B\epsilon(t)$ is distributionally equivalent to $\epsilon(t)$.

Note that it follows from (ii) of (TZ) that the determinant of B is 1 or -1, which can be readily seen by equating the second moment of $\epsilon(t)$ and that of $B\epsilon(t)$.

Recently, Chan and Ho (2004, Theorems 3, 4, and 7) extended the unique representation result for univariate non-Gaussian linear processes to the multivariate case, which now depends on the contemporaneous dependence structure of the innovations. To state their results, we need to introduce some notation. Below, π denotes some permutation of the set $\{1, 2, \cdots, q\}$ where i is permuted to $\pi(i)$. We write the ith component of $\epsilon(t)$ as $\epsilon_i(t)$ and the ith column of $A(j)$ as $A_i(j)$. Under the condition

(C1) the innovations consist of independent and identically distributed components,

they showed that the two linear representations (1) and (2) are related by the equations (for all t)

$$\epsilon'_i(t - m(i)) = b_i \epsilon_{\pi(i)}(t), \quad \text{and} \quad A_{\pi(i)}(t - m(i)) = A'_i(t) b_i. \tag{4}$$

for some scalars b_i, integers $m(i), i = 1, 2, \cdots, q$ and permutation π of the set $\{1, 2, \cdots, m\}$. In other words, the innovations in one representation are obtained by essentially shifting the innovations in the other representations, although the shift may be component-specific. Similarly, the coefficients in one representation are obtained by shifting their counterparts in the other representation with the shift possibly column-specific. This variable shift is the reason why the TZ-condition is not necessary for time-reversibility. Chan and Ho (2004) obtained the same uniqueness result stated in (4) in the case of non-identical components under the following additional moment assumption.

(C2) The innovations have independent components and there exists an $r \geq 3$ such that each component of an innovation has non-zero rth cumulant. Also, each component admits a finite τ-moment where τ is an even integer greater than r.

Intuitively, if the innovations are strongly, contemporaneously correlated, then the shifts in (4) must be the same. Indeed, Chan and Ho (2004) showed that this is the case, i.e. there exist an integer m and a matrix B such that

$$\epsilon'(t - m) = B\epsilon(t), \quad \text{and} \quad A(t - m) = A'(t)B, \tag{5}$$

under the following conditions.

(C3) The innovation sequence ϵ admits an invertible B_K for some K with an $r \geq 3$, where $K = (k_3, k_4, ..., k_r)$ is a multi-index, $1 \leq k_i \leq q$ for all $3 \leq i \leq r$, and the matrix B_K is the $q \times q$ matrix where the (i, j)th entry of B_K is the cumulant α_{ijK} =cum$(\epsilon_i(t), \epsilon_j(t), \epsilon_{k_3}(t), ..., \epsilon_{k_r}(t))$.

(C4) Any two linear combinations of $\epsilon(t)$ with non-zero coefficients must be stochastically dependent.

Chan and Ho (2004) demonstrated that the preceding two conditions are satisfied if, e.g., the innovations are multivariate t-distributed with a positive-definite covariance matrix.

In the univariate case, Cheng (1999, Theorem 1) showed that the uniqueness of the representation for a non-Gaussian process implies that if X is given by equation (1) and

$$Y(t) = \sum A'(j)\epsilon'(t-j), \tag{6}$$

then Y is distributionally equivalent to X if and only if (3) holds, with the equality concerning the innovations interpreted as equivalence in distribution. For the multivariate case, the situation is much more challenging due to the vastly richer structure for the *multivariate* innovations. Below, we shall extend Cheng's characterization of distributionally equivalent linear processes to the multivariate case.

2 MAIN RESULTS

Our first result is the aforementioned characterization of distributional equivalence of two linear processes. As it is well known that two (centered) Gaussian processes are equivalent in distribution if and only if they have identical spectrum, we shall focus on the case of non-Gaussian processes.

Theorem 1. *Let X and Y be two non-Gaussian $p-$dimensional linear processes defined by (1) and (6) respectively. Then X and Y are distributionally equivalent if and only if (4) holds under conditions (C1) or (C2), or (5) holds under conditions (C3) and (C4), with equalities concerning the innovations interpreted as equivalence in distribution.*

The proof of this characterization result is given in an appendix.

Recent results on the uniqueness of the linear representation of a multivariate non-Gaussian linear process (and the above characterization of distributionally equivalent multivariate non-Gaussian processes) enable us to derive necessary and sufficient conditions for time-reversibility, which is the main purpose of this note.

Define $Y(t) = X(-t)$. Then time-reversibility of X implies that X and Y are equivalent in distribution. To state our results characterizing time-reversibility for the case of independent-component innovations, we introduce the notion of a symmetric pair of columns of A. We say that the ith and jth columns of the coefficient sequence A form a symmetric pair if there exist a non-zero scalar b and an integer m such that, for all t, $A_j(-t-m) = A_i(t)b$ and $\epsilon_i(t)$ and $b\epsilon_j(t)$ have identical distribution. Notice that if $i = j$, the above notion of symmetry reduces to the requirement that $\{A_i(t), t = \cdots, -1, 0, 1, \cdots\}$ is essentially symmetric about $-m/2$. Assuming that the components of an innovation term are independent plus some other regularity conditions, the following result shows that time-reversibility holds if and only if the coefficient matrices consist of symmetric columns or symmetric pairs of columns, with column- or pair-specific origins of symmetry.

Theorem 2. *Let X be a non-Gaussian linear process defined by (1). Suppose that the innovations have independent components and either (C1) or (C2) holds. Then X is time-reversible if and only if it holds that*

(C5) *there exist scalars b_i and integers $m(i), i = 1, 2, \cdots, q$, a permutation π of $\{1, 2, \cdots, q\}$ such that (i) π is the inverse of itself, i.e. for each i, $\pi(i) = i$ or $\pi(\pi(i)) = i$, (ii) for all t, $A_{\pi(i)}(-t - m(i)) = A_i(t)b_i$ and (iii) $\epsilon_i(t) = b_i\epsilon_{\pi(i)}(t)$, in distribution.*

Note that if $\pi(i) = i$, (ii) and (iii) imply that the ith column of A is essentially symmetric about $-m(i)/2$. On the other hand, if $\pi(i) = j$ and $j \neq i$, (ii) and (iii) entail that the ith and jth columns of A form a symmetric pair. Note that $m(i)$ need not be identical and $b(i)$ must be 1 or -1 if $\pi(i) = i$. We remark that if $q = 1$, i.e. the case of univariate innovations, then (C1) trivially holds and the preceding theorem implies the aforementioned result of Tong and Zhang (2005) that condition (TZ) is necessary and sufficient for time-reversibility of X with $q = 1$.

If the innovations are contemporaneously dependent so that (C3) and (C4) hold, then the following result says that time-reversibility occurs if and only if the coefficients are essentially symmetric about some origin that is the same for each column.

Theorem 3. *Let X be a non-Gaussian linear process defined by (1). Suppose that conditions (C3) and (C4) hold. Then X is time-reversible if and only if condition (TZ) holds.*

The proof of the theorem is similar to that of Theorem 2, and hence is omitted.

Finally, we note that a multivariate linear process generally arise from a vector stochastic difference equation specified by an autoregressive-moving average model. Chan and Tong (2002) showed that a general Markov process defined by a state-space model can be equivalently expressed in terms of a vector stochastic difference equation, under some general regularity conditions. Thus, the results obtained herein may be useful for studying the time-reversibility of Markov simulation techniques.

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Appendix 1 Proof of Theorem 1.

Assume (C1) holds. The sufficiency part of the Theorem can be seen by the following equalities obtained via (4), where $\stackrel{D}{=}$ denotes equality in distribution.

$$
\begin{aligned}
Y(t) &= \sum_{j=-\infty}^{\infty} A'(j)\epsilon'(t-j) \\
&= \sum_{i=1}^{q}\sum_{j=-\infty}^{\infty} A_i'(j)\epsilon_i'(t-j) \\
&\stackrel{D}{=} \sum_{i=1}^{q}\sum_{j=-\infty}^{\infty} A_i'(j)b_i\epsilon_{\pi(i)}(t-j+m(i)) \\
&= \sum_{i=1}^{q}\sum_{j=-\infty}^{\infty} A_{\pi(i)}(j-m(i))\epsilon_{\pi(i)}(t-j+m(i)) \\
&= \sum_{i=1}^{q}\sum_{j=-\infty}^{\infty} A_{\pi(i)}(j)\epsilon_{\pi(i)}(t-j) \\
&= \sum_{j=-\infty}^{\infty} A(j)\epsilon(t-j) \\
&= X(t).
\end{aligned}
$$

We now verify the necessity part of the Theorem by adapting the proof technique of Cheng (1999, Theorem 1) to our case. Let H_X and H_Y denote the Hilbert spaces generated by X and Y respectively. As a result of the distributional equivalence of X and Y, the correspondence $X(t) \leftrightarrow Y(t)$ can be naturally extended to show that H_X and H_Y are isometric isomorphic. Because $\hat{A}$ is of full-rank almost everywhere, Lemma 2 of Chan and Ho (2004) implies that $\epsilon(t) \in H_X$. Then there exists $V(t) \in H_Y$ corresponding to $\epsilon(t) \in H_X$. As a result of the equivalence and isometric isomorphism, the Vs are independent and identically distributed with the same distribution as ϵs and $Y(t) = \sum A(j)V(t-j)$. Then

$$Y(t) = \sum A(j)V(t-j) = \sum A'(j)\epsilon'(t-j).$$

By (4), we have

$$\epsilon'_i(t - m(i)) = b_i V_{\pi(i)}(t), \quad \text{and} \quad A_{\pi(i)}(t - m(i)) = A'_i(t)b_i.$$

Because V has the same distribution as ϵ, we conclude that $\epsilon'_i(t - m(i)) = b_i\epsilon_{\pi(i)}(t)$ in distribution, which completes the proof of the necessity part of the Theorem. The proofs of the Theorem under the other regularity conditions are similar and hence are omitted.

Appendix 2 Proof of Theorem 2.

Recall that $Y(t) = X(-t) = \sum A'(j)\epsilon'(t-j)$ where $A'(j) = A(-j)$ and $\epsilon'(t) = \epsilon(-t)$, for all j and t. Time-reversibility of X holds if and only if X and Y are equivalent in distribution, which is equivalent to the validity of (4). In turn, this is equivalent to the requirement that there exist b_i, integers $m(i), i = 1, 2, \cdots, q$ and a permutation π of the set $\{1, 2, \cdots, m\}$ such that, for all t,

$$\begin{aligned} &(a) \quad \epsilon_i(-t + m(i)) = b_i\epsilon_{\pi(i)}(t), \text{ in distribution, and} \\ &(b) \quad A_{\pi(i)}(t - m(i)) = A_i(-t)b_i \end{aligned} \tag{A1}$$

Note that (a) can be re-written as $\epsilon_i(t) = b_i\epsilon_{\pi(i)}(t)$ in distribution, because ϵ consists of independent and identically distributed variables. Also, (b) is equivalent to the condition that, for all t, $A_{\pi(i)}(-t - m(i)) = A_i(t)b_i$. Thus, the sufficiency of (C5) for time-reversibility is then clear.

It remains to verify the necessity of (C5). So, suppose that (a) and (b) holds. It follows from (a) that b_i are non-zeroes. Now, it is well-known (Herstein, 1996, Theorem 3.2.2) that π can be written as a product (composition) of disjoint cycles

$$\pi = C_1C_2...C_r$$

where each cycle is of the form $C_k = (j_1j_2....j_{l(k)})$, i.e. a permutation that maps j_1 to j_2, j_2 to j_3, ..., and $j_{l(k)}$ back to j_1. Here j_i are distinct integers in $\{1, 2, ..., q\}$. For the sake of simplicity we first discuss the following two cases.

Case 1: $q = 2, \pi(1) = 2, \pi(2) = 1$. Note that π is an even permutation here. In this case (b) implies that there exist $m(i), i = 1, 2$ and for all t

$$\begin{aligned} A_1(t) &= A_2(-t - m(1))/b_1 & \text{(A2)} \\ A_2(t) &= A_1(-t - m(2))/b_2, & \text{(A3)} \end{aligned}$$

$\epsilon_1(t)$ and $b_1\epsilon_2(t)$ have identical distribution, and so do $\epsilon_2(t)$ and $b_2\epsilon_1(t)$. Therefore, $\epsilon_1(t)$ and $b_1b_2\epsilon_1(t)$ have identical distribution, entailing that $b_1b_2 = \pm 1$. Equation (A2) implies that $A_2(t) = A_1(-t - m(1))b_1$ which equals $A_2(t + m(1) - m(2)b_1b_2$, by (A3). Hence, for all t, $A_2(t) = A_2(t + 2(m(1) - m(2)))$. If $m(1) \neq m(2)$, A_2 is periodic with a positive period, but then A_2 must be zero because of the square summability of A. Because $\hat{A}$ is of full-rank almost everywhere, it is impossible for A_2 to be zero. We conclude that $m(2) = m(1) = m$ and $b_1b_2 = \pm 1$. This shows that b_1A_1 and A_2 are symmetric images of each other about $-m/2$ and ϵ_1 and $b_1\epsilon_2$ have identical distribution.

Case 2. $q = 3, \pi(1) = 2, \pi(2) = 3, \pi(3) = 1$. Note that π is an odd permutation here. In this case (b) implies

$$\begin{aligned} A_1(t) &= A_2(-t - m(1))/b_1 \\ &= A_3(t + m(1) - m(2))/b_1b_2 \\ &= A_1(-t - m(1) + m(2) - m(3))/b_1b_2b_3 \end{aligned}$$

In the same manner from conclusion (a) we get $\epsilon_1(t)$ and $\epsilon_1(-t+m(1)-m(2)+m(3))b_1b_2b_3$ having the same distribution. Similarly, the other components of $A(t)$ and $\epsilon(t)$ satisfy analogous equations.

In the general case that $\pi = C_1C_2...C_r$ where C_k are disjoint cycles, it follows from the above two cases that (a) for even cycles C_k the corresponding columns can be paired up as $A_i, A_{\pi(i)}$ so that they have the same property as in case 1, and (b) for odd cycles C_k the columns $A_{j_1}(t), ..., A_{j_{l(k)}}(t)$ are related in a similar manner as in case 2 so that they are equivalent to products of singleton cycles. Thus, conditions (i)–(iii) in the statement of the Theorem hold. This completes the proof of the necessity of (C5).

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